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Forward Guidance and Its Effectiveness: A Macro-Finance Shadow-Rate Framework

Junko Koeda and Bin Wei

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Abstract: Forward guidance provides monetary policy communication for an economy at the effective lower bound (ELB). In this paper, we consider both calendar- and outcome-based forward guidance about the timing of liftoff. We develop a novel macro-finance shadow rate term structure model by introducing unspanned macro factors and an outcome-based liftoff condition. We estimate the model using the maximum likelihood method with extended Kalman filter. Based on the estimation results, we show that outcome-based forward guidance is indeed effective and has significant monetary-easing effects on the real economy in both ELB periods of the global financial crisis (GFC) and the COVID-19 pandemic. In particular, we find that the overall impact on the unemployment rate is about 0.8 percent during both the GFC and the pandemic, but outcome-based forward guidance contributes more in the former than in the latter ELB period (about 0.30 percent versus 0.15 percent).

JEL classification: E43, E44, E52, E58

Key words: forward guidance, effective lower bound (ELB), liftoff, term structure, shadow rate, macro finance, unspanned macro factors

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Please address questions regarding content to Junko Koeda, Graduate School of Economics, Waseda University, jkoeda@waseda.jp; or Bin Wei, Research Department, Federal Reserve Bank of Atlanta, bin.wei@atl.frb.org.

1 Introduction

Forward guidance is an important tool that many central banks use to communicate with the public about the likely future course of monetary policy. It is an especially powerful policy tool for an economy at the effective lower bound (ELB)—through the central bank's commitment to keep the policy rate low, forward guidance provides stimulus to the economy (Krugman, 1998; Eggertsson and Woodford, 2003; Jung et al., 2005).

As the global financial crisis (GFC) broke out, the Federal Reserve in the United States reduced the federal funds rate target to almost zero in December 2008. The Federal Open Market Committee (FOMC) initially used *calendar-based* forward guidance to communicate the timing of liftoff between March 2009 and September 2012. For example, in September 2012, the FOMC stated

> "*[E]xceptionally low levels for the federal funds rate are likely to be warranted at least through mid-2015.* "

Starting in December 2012, the FOMC switched to *outcome-based* forward guidance, which is conditional upon the evolving economic outlook for unemployment and inflation (related to the "Evans rule" proposed in Evans, 2011). For example, in December 2012, the FOMC stated

> "*[T]his exceptionally low range for the federal funds rate will be appropriate at least as long as the unemployment rate remains above 6-1/2 percent, inflation between one and two years ahead is projected to be no more than a half percentage point above the Committee's 2 percent longer-run goal, and longer-term inflation expectations continue to be well anchored.*"

Although the specific thresholds were dropped in a March 2014 statement, outcome-based forward guidance language has continued to highlight the link of future policy tightening to tangible evidence ("data-dependence") about the state of the economy in FOMC statements.[1] The Bank of Japan has used similar forward guidance with an explicit threshold condition,[2] and several other major central banks have emphasized on the conditionality in their forward guidance.[3]

[1]For example, following the onset of the pandemic, the FOMC statement in September 2020 indicated that "*The Committee expects to maintain an accommodative stance of monetary policy until these outcomes are achieved.*"See Table A-1 in Appendix A for more examples of outcome-based forward guidance in recent FOMC statements.

[2]For a discussion on the Bank of Japan's earlier forward guidance, known as the "policy duration effect," see Ueda (2012). Additional related discussions can be found in Koeda and Wei (2023).

[3]For example, the Bank of Canada's statement referred to its guidance as its "conditional commitment"

How effective is outcome-based forward guidance? How can we measure the monetary easing effect of forward guidance *alone*? Answering these questions is challenging because forward guidance is often used at the ELB in conjunction with other unconventional policy tools such as large-scale asset purchases, which makes it difficult to assess the monetary policy stance attributable to forward guidance alone. Moreover, forward guidance is effective insofar as the central bank's commitment is credible. With imperfect credibility of the central bank, forward guidance about the path of future rates may be misinterpreted by the private sector as a reflection of poor economic fundamentals.

In this paper, we overcome the challenges and develop a macro-finance shadow-rate term structure framework to examine the aforementioned questions. For an economy away from the ELB, its policy rate is a good indicator of the overall monetary policy stance. However, when an economy is at the ELB, its policy rate is constrained at the lower bound and is no longer informative about how accommodative the (unconventional) monetary policy is. Instead, a "shadow short rate," first introduced by Black (1995), is estimated in standard shadow rate term structure models (SRTSMs) to assess the stance of monetary policy for an economy at the ELB (e.g., Krippner, 2013b; Christensen and Rudebusch, 2015; Ichiue and Ueno, 2015; Wu and Xia, 2016).

Built on the main insight in Black (1995) that holding physical currency is always an option, particularly when interest rates become negative, the observed short rate, r_t, in standard SRTSMs is the greater between the unobserved "shadow rate", s_t, that can take positive or negative values, and the lower bound, $\underline{r}$. That is, $r_t = \max(s_t, \underline{r})$. When the shadow rate is more negative and thus further away from the ELB, the economy is expected to spend more time at the ELB (i.e., *calendar-based* forward guidance), and the monetary policy is more accommodative.

However, as in the examples of FOMC statements above, standard SRTSMs are silent about *outcome-based* forward guidance, which links the timing of liftoff partly to the underlying macroeconomic conditions. As a major contribution of this paper, we extend the standard SRTSMs to include key macroeconomic factors and to explicitly incorporate the outcome-based forward guidance.

First, we extend the SRTSMs by including key macroeconomic factors, namely, GDP growth (GRO) and inflation (INF). We refer to the extended macro-finance shadow-rate term structure model as "*MFSRTSM0*." Following Joslin et al. (2014), we assume the macro factors are "unspanned" by yield curve factors, which plays an important role in keeping our

and stated that the next policy move was "conditional on the outlook for inflation." As another example, the Riksbank's statement emphasized that "the future direction for monetary policy... depend[ed on]... the prospects for inflation and economic activity." See Table A-2 in Appendix A for more examples from policy statements by these central banks.

models tractable.

A major benefit of including unspanned macro factors in the MFSRTSM0 is to mitigate omitted-variable biases in estimating the persistence of yield curve factors that would otherwise arise in standard yields-only SRTSMs. For example, the level or slope of the yield curve is usually positively correlated with GDP growth or inflation. Excluding macro variables would make standard SRTSMs susceptible to a positive omitted-variable bias in the estimate of the persistence of the level or slope factor. We show that the presence of the macro factors in our MFSRTSM0 helps mitigate positive bias in the persistence of the shadow rate, rendering it less persistent (or subject to a larger degree of mean reversion). As a result, the shadow rate in the MFSRTSM0 is typically higher (i.e., less negative) than that in standard SRTSMs during an ELB period. Based on our estimation results, we find that the shadow rate under the standard SRTSM achieves its minimal values of -4.78% in 2014Q1 and -4.85% in 2021Q1, while the shadow rate under the MFSRTSM0 is -4.03% and -3.33%, respectively.

It is worth mentioning that the approximation of forward rates in the standard SRTSMs in Krippner (2013b) or Wu and Xia (2016) continues to hold even in the MFSRTSM0 with the unspanned macro factors (see Proposition 1 in Section 3.1). It is therefore not surprising that the empirical evidence in the next section finds that the shadow rate estimated from the standard Wu–Xia model is a key driver for near-dated forward rates. However, the empirical evidence also suggests that macro factors such as GRO and INF have significant explanatory power in addition to the shadow rate during the ELB period of the financial crisis (see Table 1 in Section 2). For example, consider the "in-one-year-for-one-quarter" forward rate. In the pre-ELB period between 1990 and 2008, the R^2 of the regression of the forward rate on the lagged shadow rate alone is about 73%, while adding the macro factors only slightly increases the R^2 to 76%. In contrast, in the ELB period between 2009 and 2015, adding the macro factors significantly increases the R^2 from 13% to 59%. We argue that the significant increase in the macro factors' explanatory power during the ELB period of the financial crisis is attributable to the absence of outcome-based forward guidance in the MFSRTSM0.

Therefore, we next extend the MFSRTSM0 further to introduce explicitly outcome-based forward guidance. We refer to the resulting model as "*MFSRTSM*," which is essentially the model of MFSRTSM0 plus outcome-based forward guidance. In the spirit of FOMC statements with an outcome-based liftoff condition, we assume that liftoff will not occur in the model until two conditions are both satisfied: the shadow rate itself must exceed the ELB, *and* a certain macroeconomic index, assumed to be an equal-weighted average of GRO and INF in the model, exceeds a prespecified threshold. The former condition captures calendar-based forward guidance as in the SRTSM or MFSRTSM0, while the latter condition captures outcome-based forward guidance in the MFSRTSM.

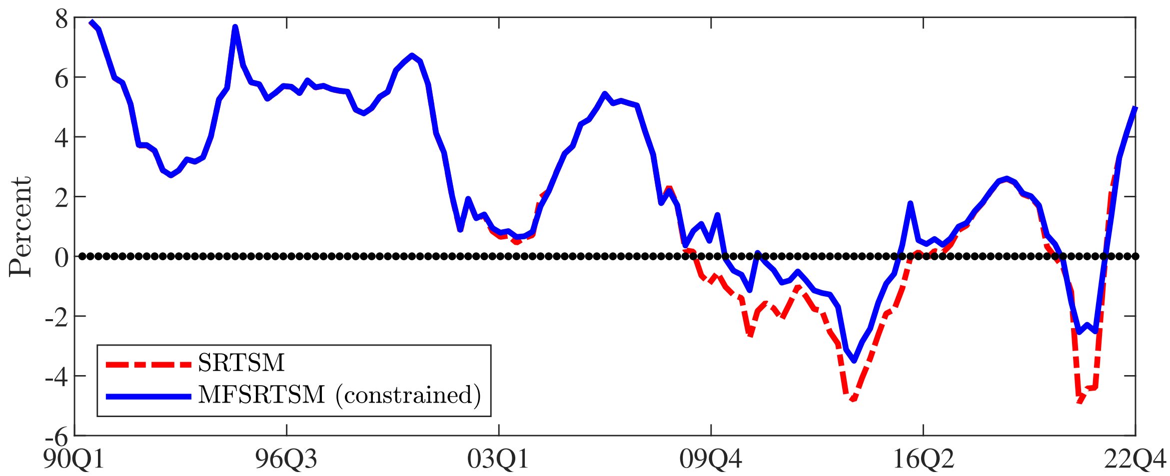
Relative to the standard SRTSMs, the ultimate model in this paper, MFSRTSM, features both unspanned macro factors and outcome-based forward guidance. The model remains tractable so that we can derive an analytical formula to approximate forward rates, which are now approximated by a highly non-linear function of both yield curve and macro factors (see Proposition 2 in Section 3.2). The dependence on macro factors arises from the outcome-based forward guidance. At normal times, when the economy is away from the ELB, the outcome-based forward guidance no longer plays a role. Thus, forward rates only depend on yield curve factors. In contrast, during ELB periods, the outcome-based forward guidance becomes operative and the macro factors start to play a vital role in driving forward rates through their impact on the expectations of future short rates. Therefore, the MFSRTSM is consistent with the empirical findings discussed above and in the next section.

As another significant finding, we show that the shadow rate in the MFSRTSM needs to be higher (less negative) than in the MFSRTSM0 to match forward rates observed in the data. We further argue that the difference in the shadow rates between the MFSRTSM and MFSRTSM0 models measures the effectiveness of outcome-based forward guidance. The intuition is the following. Consider a special case where the macro factors are independent of yield curve factors. In this case, we show that the approximation of forward rates in the MFSRTSM is almost identical to that in the standard SRTSMs, except for an additional term that is the probability of the outcome-based liftoff condition being satisfied in the future (see Corollary 3 in Section 3.3). This additional term appears because the presence of the outcome-based liftoff condition makes it harder for the economy to get out of the ELB. As a result, forward rates would have to be lower to reflect the less likelihood of liftoff, all else equal. To match the data, the estimate of the shadow rate would have to be higher (i.e., less negative) in the MFSRTSM than it otherwise would be in the absence of the outcome-based liftoff condition in the MFSRTSM0.

Therefore, by comparing shadow rates under the MFSRTSM and MFSRTSM0 models, we can measure the effectiveness of outcome-based forward guidance. Put differently, if outcome-based forward guidance is ineffective (e.g., the additional liftoff condition is ignored by market participants), then the difference between the shadow rates under the MFSRTSM versus MFSRTSM0 would be negligible. On the contrary, an effective outcome-based forward guidance would drive a wedge between those shadow rates, particularly making the shadow rate higher in the MFSRTSM than in the MFSRTSM0. Based on our estimation results, we find that the shadow rate under the MFSRTSM increases to -2.65% in 2014Q1 and -1.94% in 2021Q1, respectively, implying another 29% increase in the shadow rate from that in the

MFSRTSM0, relative to the SRTSM-implied shadow rate. Thus, the significant increase in the shadow rate in the MFSRTSM strongly suggests the effectiveness of outcome-based forward guidance.

FIGURE 1: Shadow Rates for the United States between 1990 and 2022



NOTE: The solid blue line shows the shadow rate under the SRTSM, while the dashed red line shows the shadow rate under the MFSRTSM in the constrained special case where the macro and yield curve factors are constrained to be independent. The sample period is 1990Q1 through 2022Q4.

The effectiveness of outcome-based forward guidance is easier to visualize in the aforementioned special case where the macro factors are constrained to be independent of yield curve factors. In this case, the shadow rates are identical in SRTSM and MFSRTSM0 in the absence of other confounding effects of the macro factors. Figure 1 plots the time series of the shadow rates estimated from the SRTSM or MFSRTSM0 (solid blue line) and MFSRTSM (dashed red line) in the constrained special case above. Both shadow rates coincide with the policy rate during the non-ELB period until the end of 2008 when the Fed lowered the target federal funds rate essentially to zero. Since 2009, the SRTSM-implied shadow rate turns negative and drops to the lowest level of around -4% in the end of 2013, indicating a very accommodative monetary policy stance. By contrast, once we explicitly control for forward guidance in MFSRTSM, the resulting shadow rate (the dashed red line) is substantially higher. This shadow rate measures the stance of monetary policy tools except forward guidance. The gap between these two shadow rates — our measure of the effectiveness of outcome-based forward guidance — suggests that about 30% to 50% of the shadow rate implied in the standard SRTSM in 2013 and 2014 is attributable to outcome-based forward guidance alone.[4]

[4]One concern with SRTSM models is that the estimated shadow rate is sensitive to model specifications

Finally, we measure the overall stance of the unconventional monetary policy, and propose a novel method to decompose the overall monetary policy stance into different components corresponding to calendar- versus outcome-based forward guidance. Following Wu and Xia (2016), we use a factor-augmented vector autoregression (FAVAR) to measure the effects of the unconventional monetary policy on the real economy at the ELB. Specifically, we extract the first three principal components of the observed macroeconomic variables between 1960 and 2022 and use them together with the shadow rate under the standard SRTSM to identify the monetary policy shock in the same manner as in Bernanke et al. (2005).

Similar to the result in Wu and Xia (2016), we find that the unconventional monetary policy measures used by the Fed since 2009Q2 have succeeded in lowering the unemployment rate by 0.8% by 2013Q4. Moreover,we find a similar real effect of the unconventional monetary policy measures implemented since 2020Q1 on the unemployment rate. By 2021Q4 during the pandemic, the effects of the unconventional monetary policy measures were evident in a lowered unemployment rate.

Using the estimated shadow rates under both the MFSRTSM0 and MFSRTSM, we can further decompose the above effects on the unemployment rate to different components, particularly the component corresponding to outcome-based forward guidance. The decomposition is conducted by two counterfactuals. In *Counterfactual I* (or *Counterfactual II*), we consider what would have happened to the real economy if the shadow rate had matched that under the MFSRTSM0 (or MFSRTSM).

Consider first the decomposition results for the GFC. We find that raising the shadow rate to the MFSRSTM0-implied shadow rate would have increased the unemployment rate by 0.16% (Counterfactual I). A further increase to match the MFSRTSM-implied shadow rate would have raised the unemployment rate by another 0.27% (Counterfactual II). Put differently, out of the overall impact of 0.8% on the unemployment rate by 2013Q4, about 20% of which is due to the reduction in omitted-variable bias from including informative macro factors (Counterfactual I), 34% was due to the outcome-based forward guidance (Counterfactual II), and the rest, 46%, was due to the calendar-based forward guidance and other unconventional monetary policy measures.

Regarding the decomposition results for the pandemic, although the overall impact on the unemployment rate is almost the same, or 0.8% by 2021Q4, the decomposition results

and the data used for estimation (Christensen and Rudebusch, 2015; Bauer and Rudebusch, 2016; Krippner, 2020). However, these sensitivities are less a concern for this paper because we measure the effectiveness of outcome-based forward guidance through the *difference* between two versions of macro-finance shadow-rate models that have the same specification, except that the forward guidance is incorporated into one model but not the other. As a robustness check, we show that our results remain similar when we set the lower bound to be 17 basis points following Krippner (2020).

suggest a much smaller effect for the outcome-based threshold, which accounts for only 18% of the overall impact. The smaller effect of the outcome-based forward guidance during the pandemic is intuitive because of the rapid recovery of the economy and the rapid rise of inflation.

This paper is at the intersection of several strands of the literature. First, in the literature on shadow-rate term structure modeling, several papers attempt to estimate shadow-rate term structure models in the Black (1995) framework (e.g., Bomfim (2003), Krippner (2013b), Bauer and Rudebusch (2016), Wu and Xia (2016) for the US case; Ueno et al. (2006), Kim and Singleton (2012), Krippner (2013a), Ichiue and Ueno (2015) for the Japanese case.) As the Black framework has no closed-form bond pricing formula, estimating shadow rate term structure models within the framework is typically computationally intense as it requires numerical bond pricing and nonlinear filtering. Krippner (2013b) proposes an option-based methodology to provide an analytical approximation for forward rates in continuous time. Wu and Xia (2016) independently derive the equivalent analytical approximation for forward rates in a discrete-time setting and implement a three-factor specification using U.S. Treasury data.[5]

This paper is most closely related to Krippner (2013b) and Wu and Xia (2016). We extend their yields-only SRTSMs to the MFSRTSM featuring (unspanned) macro factors and outcome-based forward guidance in both continuous and discrete time. Importantly, we generalize their analytical approximation for forward dates in our extended models and propose a way to measure the impact of outcome-based forward guidance on real economic activity following the FAVAR analysis in Wu and Xia (2016).

This paper is also related to a strand of macro-finance term structure papers which explores the role of macroeconomic variables in a no-arbitrage affine framework.[6] Importantly, Joslin et al. (2014) develop a macro-finance term structure model with "unspanned" macro factors. The authors demonstrate that unspanned macro factors—meaning that principal components of Treasury bond yields have limited power in explaining their time variation—have a significant impact on bond risk premiums, particularly during the peaks and troughs of business cycles. Using the framework in Wright (2011) shows how declining inflation uncertainty may have driven down term premia in major industrialized countries between 1990

[5]Priebsch (2013) derives a second-order approximation of the Black (1995) shadow-rate model.

[6]To list a few prominent works in this vast literature: Piazzesi (2005) shows that using macroeconomic information, particularly the Federal Reserve's target, can substantially lower pricing errors; Ang and Piazzesi (2003) and Ang et al. (2004) and Ang et al. (2006) studied the joint dynamics of the macroeconomic factors and additional latent factors in a vector autoregression framework; Rudebusch and Wu (2008) developed a macro-finance specification employing a more macroeconomic structure, including a monetary policy reaction function, an output Euler equation, and an inflation equation. See Gürkaynak and Wright (2012) for an overview of the literature and references therein.

and early 2009. We use the same framework of unspanned macro factors in this paper, but with a focus on the ELB periods, showing that assuming macro factors are unspanned is crucially important in making our models (MFSRTSM0 or MFSRTSM) tractable to derive an analytical approximation of forward rates. Additionally, complementing their emphasis on the impact of unspanned macro factors on bond risk premiums, we instead emphasize their impact on the expectations of future short rates through the outcome-based liftoff condition.

With the exception of Bauer and Rudebusch (2016) and Akkaya et al. (2015), the previous two strands of studies have not developed a shadow-rate term structure model with observable macroeconomic factors, which is particularly useful during ELB periods. These two papers are the first to include macroeconomic factors into a shadow-rate model. The macro factors in their papers are *spanned* by the yield curve and the authors estimate the models using the pre-ELB period instead. Complementary to their paper, we assume the macro factors are *unspanned* by following Joslin et al. (2014) and Wright (2011) and also we derive analytical approximations for forward rates in our macro-finance shadow rate models so that we can estimate them more tractably during the entire (both ELB and non-ELB) sample period using a standard MLE methdology with the extended Kalman filter.

Lastly, this paper adds to the forward guidance literature by measuring outcome-based forward guidance and examining its monetary easing effects on the real economy. Gürkaynak et al. (2005) and Swanson (2021) propose high-frequency methods to identify the forward-guidance factor using asset price responses around each FOMC meeting and document significant persistence in this factor. Different from these papers, we explicitly model outcome-based forward guidance and identify its effect by contrasting shadow rates: one with explicit modeling of outcome-based forward guidance and the other without. As distinguished in Campbell et al. (2012), calendar- or outcome-based forward guidance is *Odyssean*, meaning that it publicly commits the policymaker to a particular course of action. The benefit of Odyssean forward guidance is to improve macroeconomic performance today through its effect on public expectations of future policy actions. This paper shares the objective of Campbell et al. (2017) in quantifying the impact of Odyssean FOMC forward guidance on macroeconomic outcomes. By identifying Odyssean forwared guidance from overnight interest rate futures rates and based on a medium-scale DSGE model,[7] the authors show that the FOMC's Odyssean guidance since the financial crisis appears to have boosted real activity.

[7]A strand of New Keynesian DSGE literature has explored the impact of forward guidance. Standard monetary models tend to overstate the impact of forward guidance on the macroeconomy—the so-called "forward guidance puzzle" (e.g., Del Negro et al., 2013; Carlstrom et al., 2015). Deviating from the standard model setting, forward guidance can be less effective in incomplete markets featuring, for example, uninsurable idiosyncratic income risk and borrowing constraints (e.g., McKay et al., 2016) and large private-sector discounting (Nakata et al., 2019).

Our paper develops a new method to identify a specific type of Odyssean forward guidance—outcome-based forward guidance—and assesses its impact on economic performance.[8]

This paper is organized as follows. In Section 2, we present some empirical findings that motivate our MFSRTSM framework. In Section 3, we develop macro-finance shadow-rate term structure models and explain the role of outcome-based forward guidance. Section 4 reports main estimation results. In Section 5 we use the FAVAR to measure the overall monetary policy stance and propose a novel way to decompose it and single out the position of forward guidance. Section 6 concludes.

2 Empirical Observations Motivating our MFSRTSM

As discussed in the introduction, outcome-based forward guidance links the timing of liftoff partly to the underlying macro conditions. As a result, outcome-based forward guidance should influence the expectations of future short rates at the ELB. On the other hand, Joslin et al. (2014) provide empirical evidence that macro risks such as GDP growth and inflation are not spanned by bond yields and account for a large portion of the variation in forward term premiums for a non-ELB period prior to the GFC. Complementing their findings, we first provide novel empirical evidence in this section for the impact of unspanned macro risk factors on expectations of future short rates at ELB periods, which helps motivate our macro-finance shadow-rate framework that is formally introduced in the following section.

Expectations of future short rates is a key determinant of forward rates, with forward risk premiums as the other determinant. If the expectations hypothesis holds, forward rates reveal market expectations of future short rates. To discern any impact of outcome-based forward guidance, we regress various forward rates on the shadow rate and macro factors such as GDP and inflation used in this paper for 2009Q1-2015Q4 ELB period and the prior non-ELB period of 1990Q1-2008Q4. Other ELB (or non-ELB) periods exist, such as 2020Q1-2021Q4 during the pandemic. Because these periods are too short to conduct a meaningful regression analysis, we focus on the two periods mentioned above.

Fixing a particular subperiod (pre-ELB or ELB), we consider two regression specifications. In specification (I), we regress forward rates only on the shadow rate, which is estimated from the standard SRTSM as in Wu and Xia (2016). In specification (II), we augment the shadow rate with two more regressors: GDP growth and the inflation rate. We consider 14 different "in-$n$years-for-one-quarter" forward rates, where we choose n to be

[8] Another type of forward guidance—"Delphic"—publicly forecasts macroeconomic performance and/or policy actions based on the policymaker's potential superior information, but promises nothing. Delphic forward guidance improves macroeconomic outcomes by reducing private decisionmakers' uncertainty (see Campbell et al., 2012, for empirical evidence)

Table 1: Regressions of forward rates on the shadow rate and macro factors

Dep. Var	pre-ELB (I)	pre-ELB (II)	ELB (I)	ELB (II)
1Q-1Q forward	87.7	89.6	51.0	57.7
2Q-1Q forward	83.0	85.3	42.2	63.6
3Q-1Q forward	78.1	80.7	25.7	61.3
4Q-1Q forward	73.1	76.1	13.4	59.4
6Q-1Q forward	63.2	67.8	2.6	58.7
2Y-1Q forward	54.2	61.4	0.3	59.3
3Y-1Q forward	39.7	53.6	0.1	58.7
4Y-1Q forward	29.2	49.5	0.6	55.9
5Y-1Q forward	22.0	47.1	2.6	52.6
6Y-1Q forward	17.0	45.4	5.7	49.5
7Y-1Q forward	13.6	44.0	9.2	46.6
8Y-1Q forward	11.4	42.6	12.3	43.9
9Y-1Q forward	10.0	41.2	14.6	41.1
10Y-1Q forward	9.1	39.8	16.1	38.1

Note: Each row reports R^2s of regressions in percentage using forward rates as the dependent variable (indicated in the first column) with specifications (I) and (II) in Columns "(I)" and "(II)", respectively. In regressions with specification (I), The forward rate is regressed on the Wu-Xia shadow rate only under specification (I), but on both the shadow rate and two macro factors, GDP growth rate and CPI inflation rate under specification (II). We run the regressions separately for two subsample periods, namely the "pre-ELB" period between 1990Q1 and 2008Q4, and the "ELB" period between 2009Q1 and 2015Q4.

0.25, 0.5, 0.75, 1, 1.5, and 2 through 10 years. We report the R^2s of those regressions in Table 1.

Consider first the pre-ELB period prior to the GFC. If we use forward rates for no more than 2 or 3 years from now, including macro factors in specification (II) generates almost the same R^2s as those in specification (I) with the Wu–Xia shadow rate only (see the second and third columns of Table 1). These results suggest that the shadow rate is the dominant driver of the variation in those near-dated forward rates, although its explanatory power declines gradually as forward rates are extended further into the future. In contrast, for forward rates dated for 3 years or longer from now, unspanned macro factors start to become the dominant determinants, suggesting their key role in risk premiums as shown in Joslin et al. (2014).

We turn next to the ELB period between 2009Q1 and 2015Q4. For far-dated forward rates, the regression R^2s in the last two columns for the ELB period are similar to those in the second and third columns for the pre-ELB period, which makes sense as unspanned macro factors continue to be an important driver of risk premiums.

However, a stark contrast exists between the results for near-dated forward rates for the ELB period and those for the pre-ELB period. Specifically, for forward rates for no more than 2 or 3 years from now, the regression R^2s drop rapidly and include macro factors can significantly increase the explanatory power, as shown in the last two columns of Table 1.

The above results suggest that macro factors have limited role in affecting market expectations of future short rates during the pre-ELB period, but their role seemed to have become substantially more important during the non-ELB period. We argue that it is the newly implemented outcome-based forward guidance that assigns a new key role for macro factors during the ELB period. To illustrate this idea, we turn to our new theoretical framework in the next section.

3 Macro-Finance Shadow Rate Models

In this section, we develop various macro-finance shadow-rate term structure models to illustrate the effect of outcome-based forward guidance. In the first section, we extend the standard shadow-rate model (SRTSM) by incorporating unspanned macro factors and refer to the extended model as MFSRTSM0. The extended model, MFSRTSM0, remains tractable and has the same formula for instantaneous forward rates as those in SRTSM. In the second section, we further extend MFSRTSM0 to include an endogenous outcome-based liftoff condition. The extended model, referred to as MFSRTSM, remains highly tractable and has a generalized analytical formula for instantaneous forward rates that depend on both bond and macro factors.

3.1 MFSRTSM0: Incorporating Unspanned Macro Factors

In MFSRTSM0, there exist n_X latent yield curve factors X_t and n_M observable macro factors M_t. Let $Z_t = (X_t', M_t')'$ be the $n_Z \times 1$ vector of state variables with $n_Z = n_X + n_M$. Following Joslin et al. (2014) and Wright (2011), the macro factors are assumed to be "unspanned". As in standard shadow rate models (Black, 1995; Krippner, 2013b; Wu and Xia, 2016), the short rate remains at the effective lower bound $\underline{r}$ until the shadow rate s_t rises and hits the bound from below, that is,

$$r_t = \max\left(s_t, \underline{r}\right), \tag{1}$$

where the shadow rate s_t is an affine function of bond factors only

$$s_t = \delta_0 + \delta_1' Z_t = \delta_0 + \delta_{1,1}' X_t + \delta_{1,2}' M_t. \tag{2}$$

The state vector Z_t follows the Ornstein–Uhlenbeck process below under the physical measure $\mathbb{P}$,

$$dZ_t = \kappa \left(\theta - Z_t \right) dt + \sigma dW_t, \tag{3}$$

where θ is a $n_Z \times 1$ vector representing the long-run level of Z_t, κ as a $n_Z \times n_Z$ matrix governs the rate of mean reversion, σ is a constant $n_Z \times n_Z$ diffusion matrix, and dW_t is a n_Z-dimensional Wiener process. We partition them with respect to the bond and macro factors as follows:

$$\kappa = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}, \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, dW_t = \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \end{bmatrix}.$$

The market prices of risk are linear with respect to the state variables,

$$\Lambda_t = \Gamma_0 + \Gamma_1 Z_t. \tag{4}$$

The resulting risk-adjusted process for Z_t under the risk-neutral measure $\mathbb{Q}$ is given by

$$dZ_t = \kappa^* \left(\theta^* - Z_t \right) dt + \sigma dW_t^*, \tag{5}$$

where $\kappa^* = \kappa + \sigma \Gamma_1$ and $\theta^* = \kappa^{*-1} \left(\kappa\theta - \sigma\Gamma_0 \right)$. The parameters κ^* and θ^* and the Wiener process dW_t^* have the same partition as their measure-$\mathbb{P}$ counterparts.

Joslin et al. (2014) present strong empirical evidence that macro variables such as output growth and inflation cannot be completely replicated or spanned by portfolios of bond yields. Conceptually, the variation in bond yields can mostly be explained by a small number of bond risk factors (e.g., the first three principal components). In contrast, macro variables reflect a high-dimensional set of macro risks and thus are unlikely to be spanned by bond yields. Following Joslin et al. (2014), we impose the following assumption:

Assumption 1. ("*unspanned macro factors*") We assume that macro factors M_t are unspanned by the bond yield factors X_t. Specifically, we assume an autonomous process for the bond yield factors X_t under measure $\mathbb{Q}$ (i.e., the upper-right block of κ^* is zero) and the shadow rate does not directly depend on M_t (i.e., the last n_M elements of δ_1, or $\delta_{1,2}$, are zero). That is,

$$\kappa_{12}^{*} = 0, \tag{6}$$

$$\delta_{1,2} = 0. \tag{7}$$

Moreover, by applying Proposition 1 in Joslin et al. (2011) to the latent bond factors X_t, we can focus on the canonical representation where κ_{11}^{*} is in ordered real Jordan form, $\sigma_{12}=0$, $\theta_1^{*}=0$, and $\delta_{1,1}=\iota$ is a $n_X \times 1$ vector of ones.

Next, we specify the market prices of risk Λ_t. For the model to be as parsimonious as possible, we assume the market prices of *macro* risk are zero; that is, the last n_M rows of Λ_t are zero. Let λ_t denote the first n_X rows of Λ_t, which is a $n_X \times n_Z$ matrix representing the market prices of bond risk and can be written as

$$\lambda_t = \lambda_0 + \lambda_1 Z_t = \lambda_0 + \lambda_{1,1} X_t + \lambda_{1,2} M_t, \tag{8}$$

with λ_0 and λ_1 being the first n_X rows of Γ_0 and Γ_1, respectively. The conditions (6-7) together with the canonical representation impose restrictions on the parameters.[9]

First consider the Gaussian term structure model (GTSM), where there is no ELB and the short rate is simply equal to the shadow rate, i.e., $r_t = s_t = \delta_0 + \delta_{1,1}' X_t$. It is well known that the price of a maturity-τ bond in GTSM is an affine function as follows:[10]

$$P_{t,\tau}^{GTSM} = \exp\left(-A_\tau - B_\tau' X_t\right),$$

where $A_\tau = -\delta_0 \tau + \frac{1}{2}\int_0^\tau tr\left(B_u' \sigma_{11}\sigma_{11}' B_u\right) du$ and $B_\tau = -\left(I - \exp\left(-\kappa_{11}^{*\prime}\tau\right)\right)\left(\kappa_{11}^{*\prime}\right)^{-1}\iota$. The instantaneous forward rate in GTSM is

$$f_{t,\tau}^{GTSM} = -\frac{\partial \log P_{t,\tau}^{GTSM}}{\partial \tau} = -\frac{dA_\tau}{d\tau} - \frac{dB_\tau'}{d\tau} X_t \equiv a_\tau + b_\tau' X_t, \tag{9}$$

where $a_\tau = \delta_0 - \frac{1}{2} tr\left(B_\tau' \sigma_{11}\sigma_{11}' B_\tau\right)$ and $b_\tau = \exp\left(-\kappa_{11}^{*\prime}\tau\right)\iota$.

In contrast, bond prices and forward rates in MFSRTSM0 have no closed-form formula once we impose the ELB that prevents the short rate from going below the lower bound. We follow instead the methodology in Krippner (2013b) and derive a tractable approximate

[9]Specifically, the resulting parameter restrictions are $\lambda_0 = \sigma_{11}^{-1}\left(\kappa_{11}\theta_1 + \kappa_{12}\theta_2\right)$, $\lambda_{1,2} = -\sigma_{11}^{-1}\kappa_{12}$, $\theta_2^{*} = \left(\kappa_{22}^{*}\right)^{-1}\left(\kappa_{21}\theta_1 + \kappa_{22}\theta_2 - \sigma_{21}\lambda_0\right)$, $\kappa_{11}^{*} = \kappa_{11} + \sigma_{11}\lambda_{1,1}$, and $\kappa_{21}^{*} = \kappa_{21} + \sigma_{21}\lambda_{1,1}$,$\kappa_{22}^{*} = \kappa_{22} + \sigma_{21}\lambda_{1,2} = \kappa_{22} - \sigma_{21}\sigma_{11}^{-1}\kappa_{12}$.

[10]The expressions for the coefficients A_τ and B_τ are derived from the standard no-arbitrage pricing equations: $\frac{dA_\tau}{d\tau} = \frac{1}{2} tr\left(B_\tau' \sigma_{11}\sigma_{11}' B_\tau\right) - \delta_0$ and $\frac{dB_\tau}{d\tau} = -\kappa_{11}^{*\prime} B_\tau - \delta_{1,1}$ with the initial conditions $A_0 = 0$ and $B_0 = 0$.

formula in MFSRTSM0. Interestingly, the approximate formula, which is reported in the proposition below, remains the same as the formula in SRTSM without the macro factors in Krippner (2013b).

Proposition 1. *In MFSRTSM0, the instantaneous forward rate can be approximated by*

$$f_{t,\tau}^{MFSRTSM0} = \underline{r} + \sigma_\tau^{\mathbb{Q}} g\left(\frac{a_\tau + b_\tau' X_t - \underline{r}}{\sigma_\tau^{\mathbb{Q}}}\right), \tag{10}$$

where

$$g(z) \equiv z\Phi(z) + \phi(z),$$

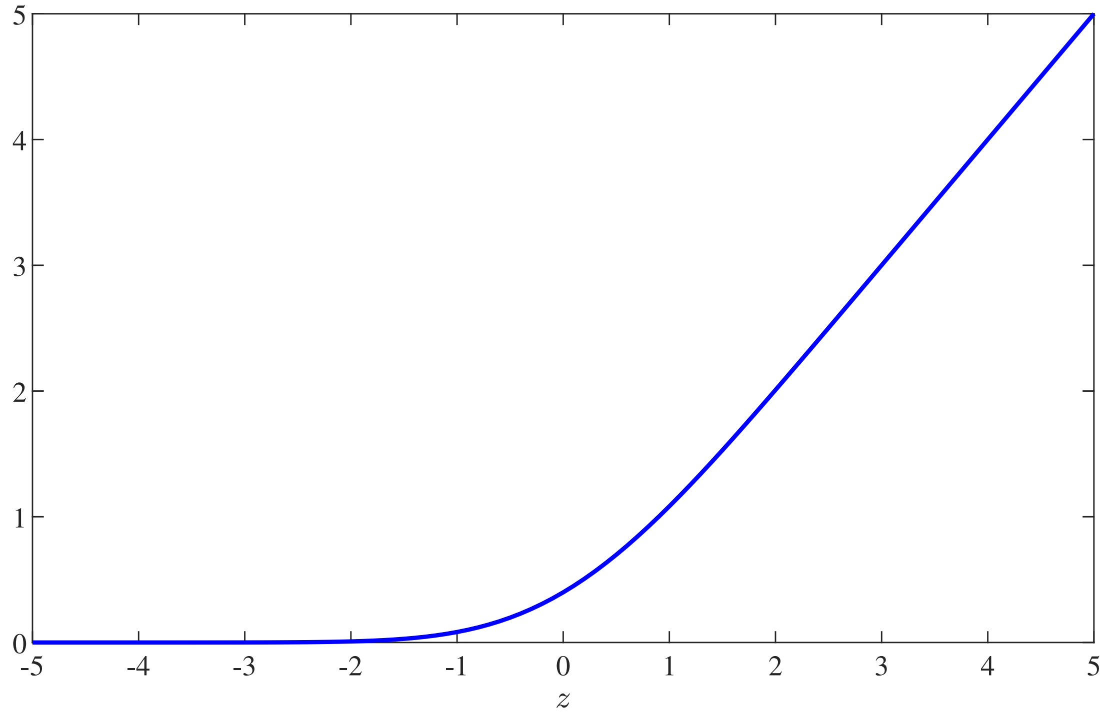
is a strictly increasing function consisting of a normal cumulative distribution function $\Phi(\cdot)$ and a normal probability density function $\phi(\cdot)$, and $a_\tau + b_\tau' X_t$ is the instantaneous forward rate in GTSM in (9).

Proof. See Appendix B. □

The following two observations are useful. First, the approximate formula for forward rates in MFSRTSM0 in (10) in Proposition 1 is essentially identical to the one in the discrete-time setting in Wu and Xia (2016) (see equation 7 on page 257 in their paper) provided that the macro factors are *unspanned*. Figure 2 plots the univariate function $g(z)$ defined in Proposition 1. On the one hand, when the shadow rate is far below the lower bound (i.e., the distance to liftoff, z, is sufficiently negative), the function value is essentially zero, meaning that the forward rate is close to the lower bound. On the other hand, when the shadow rate is sufficiently above the lower bound, the liftoff should have already occurred and the function becomes a 45-degree line, implying that in this case $g(z) \approx z$ and thus $f_{t,\tau}^{MFSRTSM0} \approx a_\tau + b_\tau' X_t$ as in standard Gaussian term structure models.

Second, the approximation formula implies that compared to yield curve factors, macro factors are not important in driving forward rates in the SRTSM or MFSRTSM0. This implication is consistent with the empirical evidence in Section 2 for the pre-ELB period between 1990 and 2008, where the shadow rate estimated from the standard Wu–Xia model was the dominant determinant of near-dated forward rates for no more than two or three years into the future. However, the above implication is *inconsistent* with the empirical evidence in the ELB period of the financial crisis between 2009 and 2015 revealing that macro factors such as GRO and INF had a significant explanatory power in addition to the shadow rate (see Table 1). We argue that the significant increase in the macro factors' explanatory power during the ELB period of the financial crisis is attributable to the absence of outcome-based forward guidance in the MFSRTSM0. In the following subsection, we extend the MFSRTSM0 by incorporating the outcome-based forward guidance.

FIGURE 2: The Univariate Function $g(\cdot)$



NOTE: This figure plots the univariate function $g(z)$ defined in Proposition 1.

3.2 MFSRTSM: Incorporating Outcome-Based Forward Guidance

In this subsection, we present our main model "MFSRTSM," in which outcome-based forward guidance is explicitly modeled by a macroeconomic liftoff condition. Under this condition, an economy at the ELB does not lift off, unless a certain macroeconomic condition is satisfied in addition to the shadow rate exceeding the ELB. When the threshold for the macroeconomic liftoff condition is sufficiently low, the condition never binds, so that MFSRTSM is reduced to MFSRTSM0.

Forward guidance has been a powerful unconventional monetary policy tool, adopted by many central banks. We explicitly model forward guidance and incorporate it into our macro-finance shadow-rate term structure model framework, referred to as the "MFSRTSM." Specifically, we focus on the type of forward guidance in the form of the following endogenous liftoff condition:

$$m_t \equiv \delta_2' M_t \geq \underline{m}, \tag{11}$$

where, as before, M_t is a vector of macro variables, δ_2 is a conformable vector of weights on the macro variables, m_t thus represents a weighted macro index, and $\underline{m}$ denotes the threshold. Through condition (11), the liftoff condition is partly tied to the macroeconomic variables. For example, if $M_t = (GDP_t; INF_t)'$ contains the GDP growth rate (GDP_t) and the inflation rate (INF_t) as used in our model estimation in the next section, $\delta_2 = (0,1)'$ assigns weighting only to inflation, such that $\underline{m}$ becomes the threshold for the inflation rate. At the other extreme, $\delta_2 = (1,0)'$ assigns weighting only to the GDP growth rate. Thus, the weightings in the vector δ_2 reflect the relative importance of macro variables in driving the monetary authority's liftoff decision.

In sum, in the general MFSRTSM, we make the following assumption about liftoff.

Assumption 2. ("*forward guidance*") We assume that a liftoff is not triggered until $s_t \geq \underline{r}$ and $m_t \geq \underline{m}$, such that

$$r_t = \underline{r}\mathbf{1}_{\{m_t<\underline{m}\}} + \max\left(\underline{r}, s_t\right)\mathbf{1}_{\{m_t\geq\underline{m}\}}, \tag{12}$$

where $\mathbf{1}_{\{.\}}$ is an indicator function that takes a value of one if the condition in the curly brackets is true, and zero otherwise.

Put differently, Assumption 2 implies that if only the shadow rate or the macro indicator exceeds their corresponding threshold values, the liftoff does not occur. The short rate thus stays at its lower bound. Otherwise, if both threshold values are reached, then liftoff occurs and the short rate coincides with the shadow rate. Another equivalent expression of the short rate is provided as

$$r_t = \begin{cases} s_t, & \text{if } m_t \geq \underline{m} \textit{ and } s_t \geq \underline{r}; \\ \underline{r}, & \text{otherwise.} \end{cases}$$

As a special case with an extremely low macro threshold value (i.e., when $\underline{m}$ is sufficiently negative), equation (12) is reduced to the same specification as in Wu and Xia (2016), where the short rate equals the maximum of the shadow rate and the effective lower bound. Moreover, the liftoff condition in (12), aligned with Hayashi and Koeda (2019), permits liftoff given a sufficiently favorable macroeconomic situation combined with a suitably positive shadow rate.

Under the "unspanning condition" in (7), the macro factors do not directly affect the shadow rate. However, the factors do have an indirect impact on the short rate as well as forward rates through the liftoff condition. In the proposition below, we develop an analytical approximation for the forward rate in MFSRTSM, making the otherwise complicated model tractable. We discuss shortly how the approximation compares to that in the SRTSM and examine implications about measuring the stance of forward guidance, one of main focuses in this paper.

Proposition 2. *In MFSRTSM, the instantaneous forward rate can be approximated by*

$$f_{t,\tau}^{MFSRTSM} = \underline{r} + \sigma_\tau^{\mathbb{Q}} g\left(\frac{a_\tau + b_\tau' X_t - \underline{r}}{\sigma_\tau^{\mathbb{Q}}}, \frac{c_\tau + d_\tau' M_t + e_\tau' X_t - \underline{m}}{\eta_\tau^{\mathbb{Q}}}; \varrho_\tau^{\mathbb{Q}}\right), \tag{13}$$

where

$$\begin{aligned}
g\left(z_1, z_2; \varrho\right) &\equiv h\left(-z_1, -z_2; \varrho\right) + \varrho h\left(-z_2, -z_1; \varrho\right) + z_1 F\left(z_1, z_2; \varrho\right), \\
h\left(z_1, z_2; \varrho\right) &\equiv \phi\left(z_1\right) \Phi\left(\frac{\varrho z_1 - z_2}{\sqrt{1-\varrho^2}}\right), \\
F\left(z_1, z_2; \varrho\right) &= \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \frac{1}{2\pi\sqrt{1-\varrho^2}} \exp\left\{-\frac{z_1^2 - 2\varrho z_1 z_2 + z_2^2}{2\left(1-\varrho^2\right)}\right\},
\end{aligned}$$

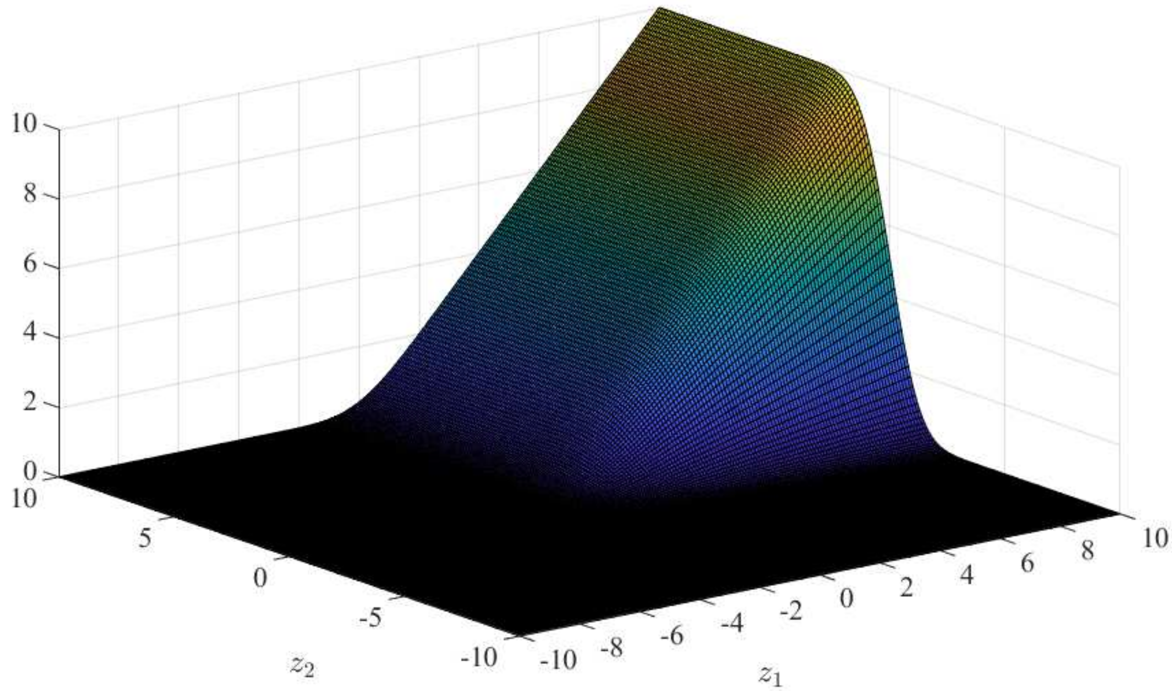
and the expressions for the coefficients c_τ, d_τ, e_τ, $\sigma_\tau^{\mathbb{Q}}$, $\eta_\tau^{\mathbb{Q}}$ and $\varrho_\tau^{\mathbb{Q}}$ are provided in equations (B-2a-c) and (B-3a-c) in the appendix.

Proof. See Appendix B. □

The approximation in (13) is a natural extension of the approximation in the MFSRTSM0 in that the univariate function $g(\cdot)$ in equation (10) is extended to a bivariate function. Figure 3 plots the function $g(\cdot,\cdot)$ with the correlation set to zero. Note that when the liftoff condition is well satisfied (i.e., z_2 is sufficiently positive), the function g with respect to the first argument z_1 takes almost the same shape as the univariate function as in Figure 2. This is intuitive, because in this case the macro index is far above the threshold $\underline{m}$ such that the outcome-base forward guidance is no longer operative. At the other extreme, when z_2 is sufficiently negative, the outcome-based forward guidance becomes so restrictive that the likelihood of liftoff is minuscule and the function is close to zero. In the most interesting case where both z_1 and z_2 take intermediate negative values, the function with respect to z_1 has a similar shape as in Figure 2, but flatter with a smaller slope.

The extension is intuitive because now the timing of liftoff in the MFSRTSM is influenced by both the yield curve factors *and* the macro factors. In the extreme case where the macro threshold $\underline{m}$ is sufficiently low, the approximation in (13) coincides with the approximation under the MFSRTSM0 or the SRTSM. In fact, in this case, the MFSRTSM reduces to the SRTSM because the macro-dependent liftoff condition is always trivially satisfied.

FIGURE 3: The Bivariate Function $g(\cdot, \cdot; \varrho = 0)$



NOTE: This figure plots the bivariate function $g(z_1, z_2; \varrho)$, which is defined in Proposition 2 with a correlation of zero (i.e., $\varrho = 0$).

3.3 Discussions

In this section, we discuss some important implications of incorporating macro factors in developing MFSRTSM0 or MFSRTSM.

The role of macro factors. Incorporating macro factors in MFSRTSM0 has two major benefits relative to the SRTSM. The first benefit lies in the fact that macro factors may contain useful information about the latent yield curve factors, especially at the ELB when the policy rate is constrained to the lower bound. Bauer and Rudebusch (2016) provides some empirical evidence for this benefit.

Another benefit is that including unspanned macro factors in the MFSRTSM0 helps mitigate omitted-variable biases in estimating the persistence of yield curve factors that would otherwise arise in standard yields-only SRTSM. For example, the level or slope of the yield curve is usually positively correlated with GDP growth or inflation. Excluding macro variables would make standard SRTSMs susceptible to a positive omitted-variable bias in the estimate of the persistence of the level or slope factor. We show that the presence of the macro factors in our MFSRTSM0 helps mitigate the positive bias in the persistence of the shadow rate, rendering the shadow rate less persistent (or a larger degree of mean reversion). As we will discuss estimation results in the next section, less-persistent shadow rate in the MFSRTSM0 would lead to a higher (less negative) shadow rate than in the SRTSM, because of the greater tendency of the shadow rate in MFSRTSM0 to mean revert to the ELB from below. Based on our estimation results, we show that the second benefit above is more important quantitatively.

The stance of forward guidance. We now compare the MFSRTSM forward rate and shadow rate with those under MFSRTSM0. As discussed below, the comparison motivates our measure of the stance of forward guidance.

To single out the effects of forward guidance only, consider the following simplified MFSRTSM where the macro factors are assumed to be orthogonal to the latent yield curve factors under the risk-neutral measure $\mathbb{Q}$; that is, $\kappa^*_{21} = \sigma_{21} = 0$ implying that κ^* and σ are both block diagonal. Corollary 3 reports the approximation of the forward rate in this simple case.

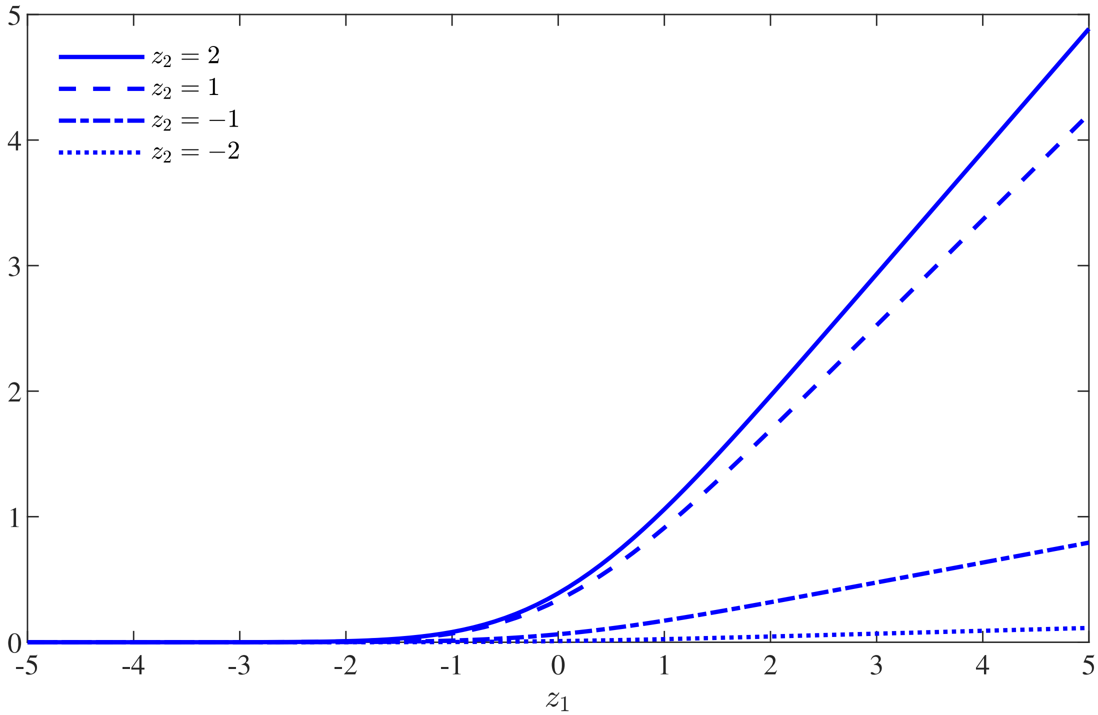
Corollary 3. *When the bond factors X_t and the macro factors M_t are* independent *of each other under the risk-neutral measure $\mathbb{Q}$ (i.e., $\kappa^*_{21} = \sigma_{21} = 0$), the approximation of the instantaneous forward rate $f^{MFSRTSM}_{t,\tau}$ in MFSRTSM can be further simplified as follows*

$$f^{MFSRTSM}_{t,\tau} = \underline{r} + \Phi\left(\frac{c_\tau + d'_\tau M_t - \underline{m}}{\eta^{\mathbb{Q}}_\tau}\right)\sigma^{\mathbb{Q}}_\tau g\left(\frac{a_\tau + b'_\tau X_t - \underline{r}}{\sigma^{\mathbb{Q}}_\tau}\right). \tag{14}$$

Proof. See Appendix B. □

The approximation of the forward rate in equation (14) under the simplified MFSRTSM is almost identical to the MFSRTSM0 counterpart in equation (10), except for the adjustment term $\Phi\left(\frac{c_\tau + d'_\tau M_t - \underline{m}}{\eta^{\mathbb{Q}}_\tau}\right)$. This term can be roughly interpreted as the probability that the macro index hits the threshold from below τ periods from now and is typically substantially less than 1 during ELB periods. As a result, to match the observed forward rates, the implied MFSRTSM shadow rate must be higher than the MFSRTSM0 shadow rate. In fact, Figure 4 plots the bivariate function $g(\cdot, \cdot; \varrho = 0)$ as a function of z_1 when the second argument z_2 is fixed at 2, 1, -1, and 2, respectively. The figure shows that as the likelihood of meeting the liftoff condition diminishes (i.e., z_2 decreases), then the function becomes less sensitive to z_1. The less sensitivity is reflected in the smaller probability $\Phi\left(\frac{c_\tau + d'_\tau M_t - \underline{m}}{\eta^{\mathbb{Q}}_\tau}\right)$ in equation (14).

FIGURE 4: The Bivariate Function $g(\cdot, z_2; \varrho = 0)$ with z_2 Fixed



NOTE: This figure plots the bivariate function $g(z_1, z_2; \varrho)$, which is defined in Proposition 2 with a correlation of zero (i.e., $\varrho = 0$).

As a result, for a given function value (equivalently, a given forward rate), it takes a larger z_1 or shadow rate to generate as the likelihood of meeting the liftoff condition diminishes.

Therefore, the shadow rate in the full MFSRTSM model has to be higher, relative to the counterpart in MFSRTSM0, as a result of the outcome-based forward guidance. Consequently, the difference between the shadow rates under MFSRTSM and MFSRTSM0 measures the stance of forward guidance only.

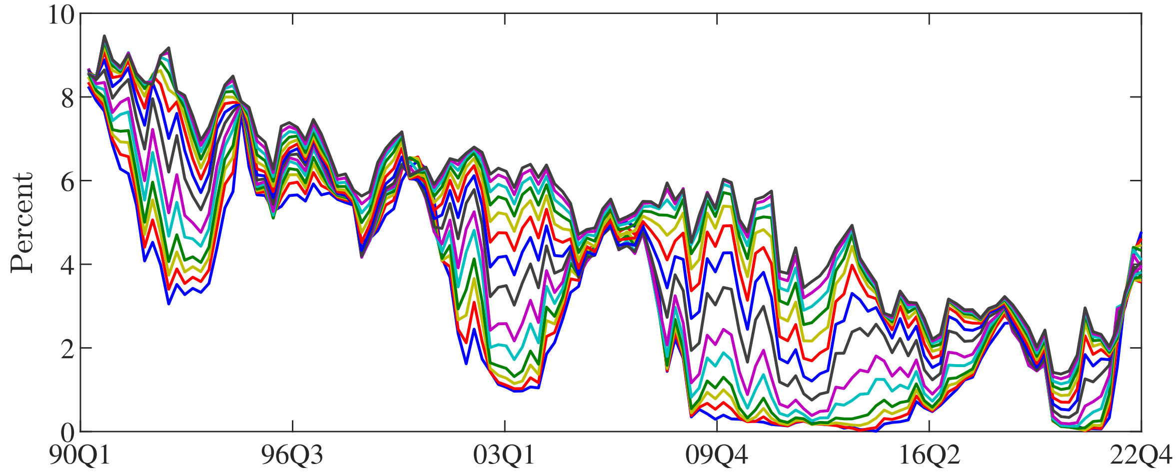
4 Estimation and Results

In this section, we first offer a brief summary of the main data used for our estimation. Next, we discuss the methodology and main results.

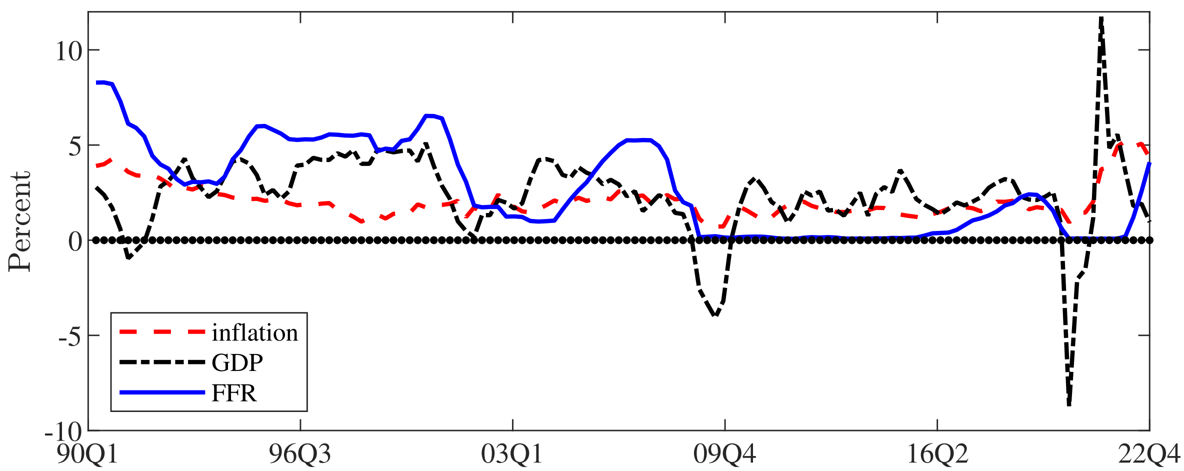
4.1 Data

We estimate our models at a quarterly frequency for the sample period from 1990Q1 to 2022Q4, using one-quarter ahead forward rates in the 1-, 2-, 3-, 4-, 6-, 12-, 16-, 20-, 24-, 28-, 32-, 36-, 40-quarter maturities obtained from Gürkaynak et al. (2007). Panel A of Figure 5 plots the historical forward rates in our sample period.

FIGURE 5: Key Data Series between 1990 and 2022



A. Forward rates



B. Macro factors and policy rate

NOTE: Panel A of this plot displays the U.S. "in-n-years-for-one-quarter" forward rates, for n equals 0.25, 0.5, 0.75, 1, 1.5, 2, ... , 10 years. The sample period is 1990Q1 through 2022Q4. Panel B of this figure plots the policy rate known as the federal funds rate (blue solid line), GDP growth (black dashed dotted line), and inflation (red dashed line).

We identify the effective lower bound $\overline{r}$ by the interest rate paid on excess reserves. The Fed began paying interest on excess reserves (of 0.25%) since January 2009 and kept the same rate until the December 2015 liftoff. Given this, we set $\overline{r}$ to 0.25%.

For the benchmark MFSRTSM estimation, we exogenously set $\underline{m}$ at 2% during the ELB periods of 2009Q1-2015Q4 and 2020Q1-2021Q4, consistent with the Fed's commitment on the inflation threshold.

For unspanned macro factors, we use the quarterly year-on-year changes using the series provided by the U.S. Bureau of Economic Analysis. For growth rate, we calculate the change in GDP from the same quarter of the previous year using quarterly real GDP data. For the inflation rate, we construct a quarterly series of inflation rates using the Personal Consumption Expenditures Price Index. Panel B of Figure 5 plots these macro factors together with the federal funds rate for our sample period.

4.2 Estimation Methodology

To estimate the models, we need to cast them in discrete time, then characterize each discrete-time model in a nonlinear state space model and estimate it using the maximum likelihood method with the extended Kalman filter. As shown below, the forward rate approximation in Proposition 1 or 2 remain almost the same in discrete-time MFSRTSM0 or MFSRTSM; in particular, the approximation in the discrete-time MFSRTSM0 has the same expression as in Wu and Xia (2016).

In discrete time, the state vector Z_t follows a first-order Gaussian vector autoregressive VAR(1) process under the physical measure $\mathbb{P}$,

$$Z_{t+1} = \mu_Z + \rho_Z Z_t + \Sigma_Z \epsilon_{Z,t+1}, \tag{15}$$

where $\epsilon_{Z,t+1} \overset{i.i.d.}{\sim} N(0, I)$ and we partition the shock vector $\epsilon_{Z,t+1}$ as well as the coefficients with respect to the yield-curve and macro factors as follows:

$$\epsilon_{Z,t+1} = \begin{bmatrix} \epsilon_{X,t+1} \\ \epsilon_{M,t+1} \end{bmatrix}, \mu_Z = \begin{bmatrix} \mu_X \\ \mu_M \end{bmatrix}, \rho_Z = \begin{bmatrix} \rho_{XX} & \rho_{XM} \\ \rho_{MX} & \rho_{MM} \end{bmatrix}, \Sigma_Z = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XM} \\ \Sigma_{MX} & \Sigma_{MM} \end{bmatrix}. \tag{16}$$

Without loss of generality, we assume Σ_Z is lower triangular (i.e., $\Sigma_{XM} = 0$) by normalization. The dynamics for the state vector Z_t under the risk neutral measure $\mathbb{Q}$ follows a similar VAR(1) process:

$$Z_{t+1} = \mu_Z^{\mathbb{Q}} + \rho_Z^{\mathbb{Q}} Z_t + \Sigma_Z \epsilon_{Z,t+1}^{\mathbb{Q}}, \tag{17}$$

where $\epsilon_{Z,t+1}^{\mathbb{Q}} \overset{i.i.d.}{\sim} N(0, I)$. The shock vector $\epsilon_{Z,t+1}^{\mathbb{Q}}$ as well as the coefficients $\mu_Z^{\mathbb{Q}}$ and $\rho_Z^{\mathbb{Q}}$ under the $\mathbb{Q}$ measure have similar partitions to their counterparts under the $\mathbb{P}$ measure.

As before, we impose the same normalization restrictions as in Joslin et al. (2011) where $\delta_0 = r_\infty^{\mathbb{Q}}$, $\delta_1 = (1, 1, 1)'$, $\mu_X^{\mathbb{Q}} = (0, 0, 0)'$ and $\rho_{XX}^{\mathbb{Q}}$ has the real Jordan form with eigenvalues in descending order, and Σ_Z is lower triangular. Under Assumption 1, the macro factors are unspanned, i.e., $\rho_{XM}^{\mathbb{Q}} = 0$.

The bond-market-specific pricing kernel $\mathcal{M}_{t+1} = \exp\left[-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\epsilon_{X,t+1}\right]$ is the projection of the economy-wide pricing kernel onto the bond market risks (i.e., X_{t+1}) and the state of the economy (i.e., Z_t), where λ_t denotes the prices of bond market risk, defined as

$$\begin{aligned}\lambda_t &= \Sigma_{XX}^{-1}\left([\mu_X + \rho_{XX}X_t + \rho_{XM}M_t] - [\mu_X^{\mathbb{Q}} + \rho_{XX}^{\mathbb{Q}}X_t]\right) \\ &\equiv \lambda_0 + \lambda_1 X_t + \lambda_2 M_t,\end{aligned}$$

where $\lambda_0 \equiv \Sigma_{XX}^{-1}(\mu_X - \mu_X^{\mathbb{Q}})$, $\lambda_1 \equiv \Sigma_{XX}^{-1}(\rho_{XX} - \rho_{XX}^{\mathbb{Q}})$, and $\lambda_2 \equiv \Sigma_{XX}^{-1}\rho_{XM}$. Similarly, as in the previous section, we assume zero prices of *macro* risk, implying that $\mu_M^{\mathbb{Q}} = \mu_M - \Sigma_{MX}\lambda_0$, $\rho_{MX}^{\mathbb{Q}} = \rho_{MX} - \Sigma_{MX}\lambda_1$, and $\rho_{MM}^{\mathbb{Q}} = \rho_{MM} - \Sigma_{MX}\lambda_2$.

We focus on the "in-n-periods-for-one-period" forward rate, $f_{t,n,n+1}$ in our discrete-time models. Proposition 4 below shows that the forward rate approximation in the discrete-time MFSRTSM has almost the same expression as their continuous-time counterpart in Proposition 2.

Proposition 4. *In the discrete-time MFSRTSM, the "in-n-periods-for-one-period" forward rate, $f_{t,n,n+1}$, can be approximated with:*

$$f_{t,n,n+1}^{MFSRTSM} \approx \underline{r} + \sigma_n^{\mathbb{Q}} g\left(\frac{a_n^{\mathbb{Q}} + b_n^{\mathbb{Q}\prime}X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}, \frac{c_n^{\mathbb{Q}} + d_n^{\mathbb{Q}\prime}M_t + e_n^{\mathbb{Q}\prime}X_t - \underline{m}}{\tau_n^{\mathbb{Q}}}; \varrho_n^{\mathbb{Q}}\right), \tag{18}$$

where the expressions for the parameters $a_n^{\mathbb{Q}}$, $b_n^{\mathbb{Q}}$, $c_n^{\mathbb{Q}}$, $d_n^{\mathbb{Q}}$, $e_n^{\mathbb{Q}}$, $\sigma_n^{\mathbb{Q}}$, $\tau_n^{\mathbb{Q}}$, and $\varrho_n^{\mathbb{Q}}$ as well as the nonlinear function $g(\cdot)$ are provided in equations (C-1a)-(C-1i) as well as (C-3) in Appendix C.

Proof. See Appendix C. □

We can then characterize each discrete-time model by a nonlinear state space system and estimate it using the maximum likelihood method with the extended Kalman filter (see Appendix D for more details).

4.3 Estimation Results

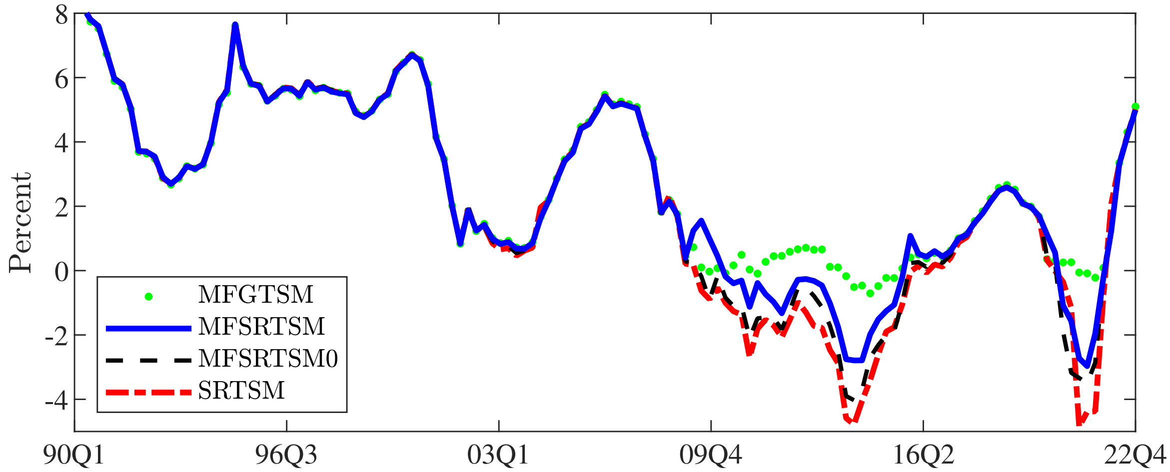
In this subsection, we report our estimation results for each model: SRTSM (shadow rate term structure model without macro factors), MFGTSM (Gaussian macro-finance term structure model without the ELB), MFSRTSM0 (macro-finance shadow rate term structure model without a liftoff condition), or MFSRTSM (macro-finance shadow rate term structure model with a liftoff condition).

Table 2-4 contain parameter estimates for SRTSM, MFGTSM, and MFSRTSM0, respectively. In the benchmark MFSRTSM estimation, we set $\delta_2 = [0.5; 0.5]$ and $\underline{m} = 2\%$ and report the estimation results in Table 5. We assign equal weights to GDP and INF to capture the notion that the outcome-based forward guidance is conditional on both inflation and economic activity. In unreported results (available upon request), we estimate δ_2 while fixing the threshold to 2% and find that $\delta_2 = [0.2; 0.8]$ has the highest likelihood value, but only slightly higher than the likelihood value under the equal-weight scheme. Furthermore, we set the threshold $\underline{m}$ to 2% to be consistent with the Fed's commitment on meeting the inflation target of 2%.

Instead of endogenously setting the threshold $\underline{m}$, we estimate the threshold as an additional parameter. The estimation results are reported in Table 6. Our results reveal that the estimated $\underline{m}$ is 1.95%, which is close to the 2% threshold we used in our benchmark MFSRTSM estimation. Recall that our full model, MFSRTSM, nests MFSRTSM0. In fact, if we set $\underline{m}$ to be sufficiently negative, then the full model reduces to the latter. Therefore, we can conduct a likelihood ratio test setting $\underline{m}$ equals to a very negative number under the null hypothesis (e.g. $H_0 : \underline{m} = -10\%$). We find that the test statistic is large, thereby rejecting MFSRTSM0 against MFSRTSM at the 1% level of significance.

Figure 6 plots the shadow rates implied by various models in this paper. First, note that in the MFGTSM, the absence of the ELB results in the estimated short rate, depicted by the green dotted line, which is close to the lower bound during the ELB periods. Similar to the estimates in Wu and Xia (2016), the shadow rate in the SRTSM (red dashed line) became as negative as -5% during both ELB periods, though it largely coincides with the observed policy rate away from the ELB. As discussed in Section 3.3, including macro factors in the MFSRTSM0 helps mitigate the positive omitted variable biases in estimating the persistence parameters in the SRTSM, thereby generating a slightly higher shadow rate (the black dashed line in the figure). Finally, incorporating outcome-based forward guidance in the MFSRTSM further accounts for the impact of the macro factors on forward rates through the liftoff condition. As a result, the resulting shadow rate in the MFSRTSM (the blue solid line) is pushed further upward compared to the rate in the MFSRTSM0. The estimates of the shadow rates implied in these models are used in the next section to assess the stance of outcome-based forward guidance.

FIGURE 6: Model-implied Shadow Rates



NOTE: This figure plots the U.S. shadow rates implied in the SRTSM (red dashed-dotted line), MFGTSM (green dotted line), MFSRTSM0 (black dashed line), and MFSRTSM (blue solid line) between 1990Q1 and 2022Q4, based on the estimation results in Tables 2-5.

5 Assessing the Stance of Forward Guidance

In this section, we use the factor augmented vector autoregression (FAVAR) model in Bernanke et al. (2005) to assess the stance of unconventional monetary policy implemented during the GFC and the pandemic. In particular, we quantify the role of outcome-based forward guidance in contributing to the overall monetary policy stance.

We extract the first three principal components of the observed macroeconomic variables between 1960 and 2022, denoted by x^m, then assume that the factors x^m and the policy rate s^o follow a VAR(4):

$$\begin{bmatrix} x_t^m \\ s_t^o \end{bmatrix} = \begin{bmatrix} \mu^x \\ \mu^s \end{bmatrix} + \rho^m \begin{bmatrix} X_{t-1}^m \\ S_{t-1}^o \end{bmatrix} + \Sigma^m \begin{bmatrix} \epsilon_t^m \\ \epsilon_t^{MP} \end{bmatrix}, \begin{bmatrix} \epsilon_t^m \\ \epsilon_t^{MP} \end{bmatrix} \sim N(0, I), \tag{19}$$

where $X_t^m = \left[x_t^m, x_{t-1}^m, \ldots, x_{t-3}^m\right]$ and $S_t^o = \left[s_t^o, s_{t-1}^o, \ldots, s_{t-3}^o\right]$. The monetary policy shock is represented by ϵ_t^{MP}, which is identified in the same way as in Bernanke et al. (2005) and Wu and Xia (2016).

Observed macroeconomic variables load on the macroeconomic factors and policy rate as follows:

$$Y_t^m = a_m + b_x x_t^m + b_s s_t^o + \eta_t^m, \eta_t^m \sim N(0, \Omega). \tag{20}$$

Note that if the ith variable in Y_t^m is denoted as $Y_t^{m,i}$ and is among the slow-moving variables (i.e., it does not respond to s_t^o), then we set $\widehat{b}_{s,i} = 0$ and regress $Y_t^{m,i}$ on a constant and x_t^m. For other variables, $Y_t^{m,i}$ is regressed on a constant, x_t^m, and s_t^o. In our estimation, we use the FRED-QD database constructed in McCracken and Ng (2020) for a large set of observed macroeconomic variables (see Appendix D for detailed information about the macroeconomic variables we use from the database).

Therefore, the contribution of monetary policy shocks between t_1 and t_2 to an individual economic variable $Y_t^{m,i}$ can be summarized by

$$\sum_{\tau=t_1}^{\min(t,t_2)} \Psi_{t-\tau}^{MP,i} \epsilon_\tau^{MP},$$

where $\Psi_j^{MP,i}$ is the impulse response

$$\Psi_j^{MP,i} = \frac{\partial Y_{t+j}^{m,i}}{\partial \epsilon_t^{MP}} = b_{x,i} \frac{\partial x_{t+j}^m}{\partial \epsilon_t^{MP}} + b_{s,i} \frac{\partial s_{t+j}^o}{\partial \epsilon_t^{MP}},$$

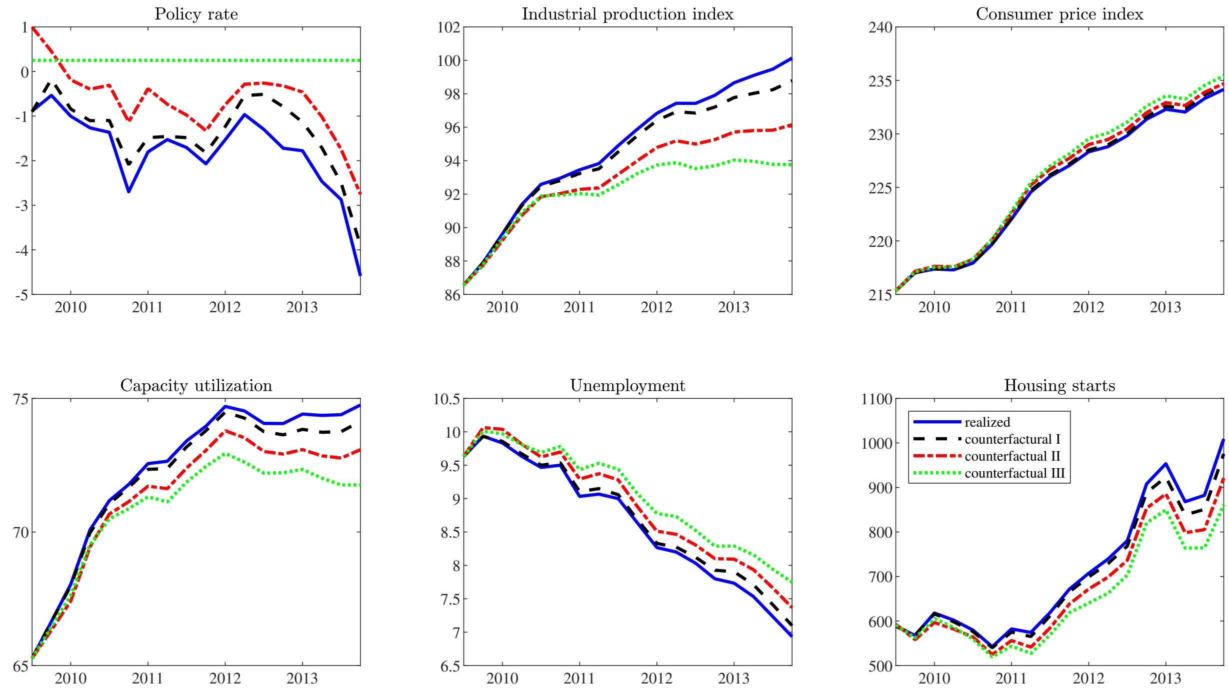
for variable i after j periods in response to a one-unit shock in ϵ_t^{MP}. The derivatives on the right-hand side are the impulse responses from a standard VAR.

In subsequent subsections, we explore counterfactual scenarios examining two periods: one post-Great Recession (from $t_1 = 2009Q3$ to $t_2 = 2013Q4$) and one during the pandemic (from $t_1 = 2020Q2$ to $t_2 = 2021Q4$). We estimate the FVAR model using the SRTSM shadow rate as the policy rate. We compute the responses of different economic variables when monetary policy shocks are aligned to produce the MFSRTSM0 shadow rate (under counterfactual I), MFSRTSM shadow rate (under counterfactual II) or a shadow rate equivalent to the ELB (under counterfactual III). Given the challenge in completely isolating shocks other than monetary policy that influence ϵ_t^{MP}, we focus on the difference in the responses rather than the level of responses.

5.1 Effect of Outcome-Based Forward Guidance during the GFC

Figure 7 presents the SRTSM shadow rate in tandem with the observed economic variables, which are shown using 'realized' blue-solid lines. When monetary policy shocks are aligned to produce the MFSRTSM0 and MFSRTSM shadow rates, the responses of these variables are illustrated under counterfactual I (black-dashed lines) and counterfactual II (red dash-dotted lines) respectively. The difference between the two counterfactuals underscores the effect of outcome-based forward guidance. Meanwhile, the disparity between the black dashed and blue solid lines signifies the macroeconomic risk premium effect.

Figure 7: Observed and Counterfactual Macroeconomic Variables during the GFC



Note: The solid blue lines show the SRTSM shadow rate (top left) and the observed economic variables between 2009Q3 and 2013Q4. The black dashed (red dotted-dashed) lines show what would have happened to these variables, if monetary policy shocks were set to generate the MFSRTSM0 (MFSRTSM) shadow rate. The green dotted lines show what would have happened if the shadow rate were kept at the lower bound.
Source: Authors' calculations.

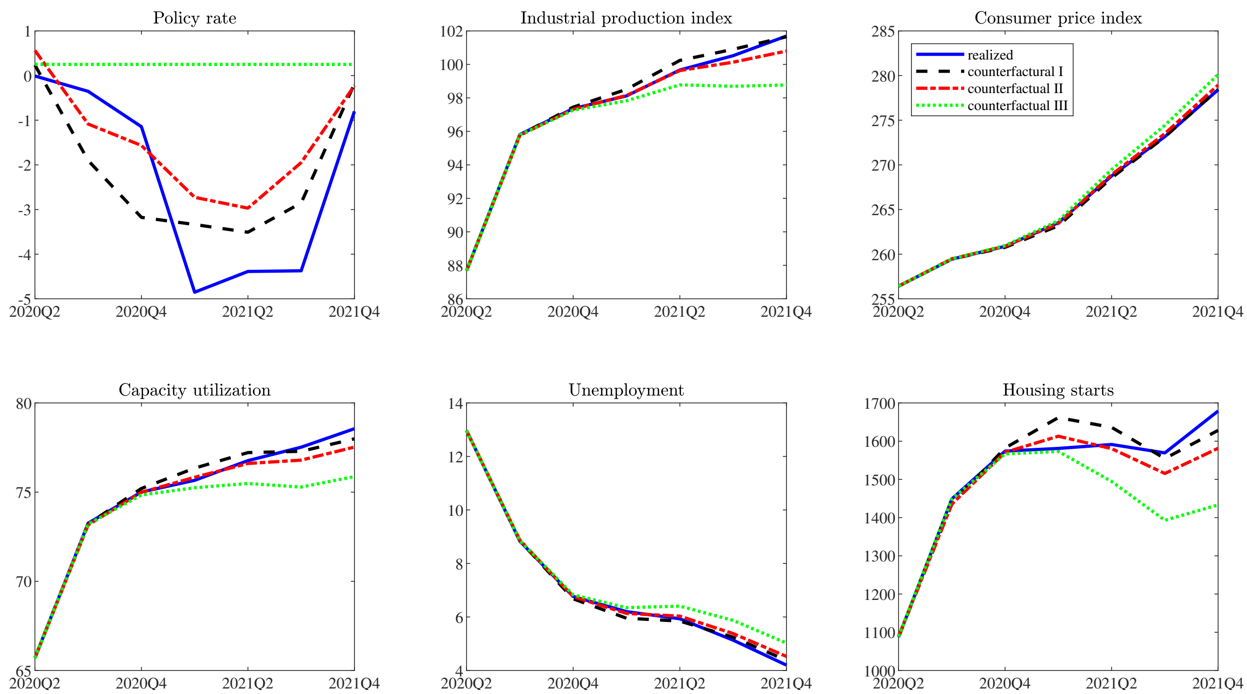
Consider the effect on the unemployment rate (see the middle-bottom panel in Figure 7). In 2013Q4, introducing unspanned macro factors (i.e., MFSRTSM0 vs SRTSM) increases the policy rate from -4.6% to -3.9% and the unemployment rate from 6.9% to 7.1%. Moreover, incorporating outcome-based forward guidance further increases the policy rate to -2.8% and the unemployment rate to 7.4%. On the other hand, if the shadow rate was kept at the lower bound of 0.25% (under counterfactual III, green dotted lines), the unemployment rate would increase to 7.7%. Thus, we can roughly decompose the overall effect of the unconventional monetary policy on the unemployment rate (or about -0.8%) into three components: -0.16% due to the risk-premium effect, -0.27% due to outcome-based forward guidance, and -0.4%due to all other unconventional policies. Relatively speaking, about one third of the overall monetary policy effect on unemployment can be explained by outcome-based forward guidance.

The results are similar for other macroeconomic variables. For example, during the GFC, the relative effect of outcome-based forward guidance is 41% for industrial production, 35%

for capacity utilization, and 37% for housing starts.

5.2 Effect of Outcome-Based Forward Guidance During the Pandemic

FIGURE 8: Observed and Counterfactual Macroeconomic Variables during the Pandemic



NOTE: Solid blue lines show the SRTSM shadow rate (top left) and the observed economic variables between 2020Q2 and 2021Q4. The black dashed (red dotted-dashed) lines show what would occur to these variables, if the monetary policy shocks were set to generate the MFSRTSM0 (MFSRTSM) shadow rate. The green dotted lines show what would have happened if the shadow rate were kept at the lower bound.
SOURCE: Authors' calculations.

Consider the impact on the unemployment rate in 2021Q4, as shown in the middle-bottom panel of Figure 8. The influence of outcome-based forward guidance on unemployment during the pandemic is less pronounced than during the GFC, due to the swift economic recovery from the pandemic. Although the overall effect of unconventional monetary policy on the unemployment rate is still around -0.8% (from 5.0% to 4.2%), only about 18% of the overall effect is attributable to outcome-based forward guidance (which decreases the unemployment rate by about 0.15%).

Similarly, we find that the relative effect of outcome-based forward guidance is smaller for other macroeconomic variables during the pandemic compared to the GFC. For example, the relative effect of outcome-based forward guidance accounts for 29% (vs. 41%) for industrial production, 17% (vs. 35%) for capacity utilization, and 19% (vs. 37%) for housing starts.

6 Conclusion

This paper provides a novel macro-finance term structure framework to analyze macro-finance linkages at and near the ELB through a state-dependent forward guidance. We apply this framework to the United States and obtain several interesting findings.

First, we argue that incorporating macro factors into standard SRTSMs help mitigate positive (omitted-variable) bias in the estimate of the persistence of yield curve factors, thereby making the shadow rate tends less negative. Put differently, failing to incorporate macro factors into a shadow rate term structure model can overestimate the ELB duration. Second, by further incorporating outcome-based forward guidance, we drive an analytical approximation formula for forward rates that generalizes the results in Krippner (2013b) and Wu and Xia (2016). In our ultimate MFSRTSM model, forward rates are approximated by a highly non-linear function of both yield curve factors and the macro factors. The dependence on macro factors arises from the outcome-based forward guidance. Third, we show that the shadow rate in the MFSRTSM needs to be higher (less negative) than in the MFSRTSM0 model without the outcome-based forward guidance. We propose to the difference in the shadow rates between the MFSRTSM and MFSRTSM0 models to measure the effectiveness of outcome-based forward guidance. Our estimation results suggest that about 30% to 50% of the shadow rate implied in the standard Wu–Xia model in 2013 and 2014 is attributable to outcome-based forward guidance alone. Lastly, employing a FAVAR-based analysis, we show that outcome-based forward guidance has significant monetary easing effects on the real economy in both ELB periods of the GFC and the pandemic. In particular, we find that the overall impact on the unemployment rate is about 0.8% during both the GFC and the pandemic, but outcome-based forward guidance contributes more in the former than in the latter ELB period (about 0.30% versus 0.15%). In a separate paper, we find similar results in the case of Japan (see Koeda and Wei (2023)).

References

Akkaya, Y., R. S. Gürkaynak, B. Kisacikoğlu, and J. H. Wright (2015): "Forward Guidance and Asset Prices," IMES Discussion Paper Series 15-E-06, Institute for Monetary and Economic Studies, Bank of Japan.

Ang, A., S. Dong, and M. Piazzesi (2004): "No-Arbitrage Taylor Rules," Working Paper, University of Chicago.

Ang, A. and M. Piazzesi (2003): "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics*, 50, 745–787.

Ang, A., M. Piazzesi, and M. Wei (2006): "What does the yield curve tell us about GDP growth?" *Journal of Econometrics*, 131, 359–403.

Bauer, M. D. and G. D. Rudebusch (2016): "Monetary Policy Expectations at the Zero Lower Bound," *Journal of Money, Credit and Banking*, 48, 1439–1465.

Bernanke, B. S., J. Boivin, and P. Eliasz (2005): "Measuring the Effects of Monetary Policy: A Factor-Augmented Vector Autoregressive (FAVAR) Approach*," *Quarterly Journal of Economics*, 120, 387–422.

Black, F. (1995): "Interest Rates as Options," *Journal of Finance*, 50, 1371–1376.

Bomfim, A. N. (2003): ""Interest Rates as Options:" Assessing the markets' view of the liquidity trap," Working Paper, Federal Reserve Board.

Campbell, J. R., C. L. Evans, J. D. M. Fisher, and A. Justiniano (2012): "Macroeconomic Effects of Federal Reserve Forward Guidance [with Comments and Discussion]," *Brookings Papers on Economic Activity*, 1–80.

Campbell, J. R., J. D. M. Fisher, A. Justiniano, and L. Melosi (2017): "Forward Guidance and Macroeconomic Outcomes since the Financial Crisis," *NBER Macroeconomics Annual*, 31, 283–357.

Carlstrom, C. T., T. S. Fuerst, and M. Paustian (2015): "Inflation and output in New Keynesian models with a transient interest rate peg," *Journal of Monetary Economics*, 76, 230–243.

Christensen, J. H. E. and G. D. Rudebusch (2015): "Estimating Shadow-Rate Term Structure Models with Near-Zero Yields," *Journal of Financial Econometrics*, 13, 226–259.

Del Negro, M., M. Giannoni, and C. Patterson (2013): "The Forward Guidance Puzzle," Federal Reserve Bank of New York Staff Report 574.

Eggertsson, G. B. and M. Woodford (2003): "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity, Economic Studies Program, The Brookings Institution*, 34, 139–235.

Evans, C. L. (2011): "The Fed's Dual Mandate Responsibilities and Challenges Facing U.S. Monetary Policy," Remarks by President Charles L. Evans at the Distinguished Speaker Seminar at European Economics and Financial Centre, London, U.K. [Accessed: 2023 08 23].

Gürkaynak, R. S., B. Sack, and E. T. Swanson (2005): "Do Actions Speak Louder Than Words? The Response of Asset Prices to Monetary Policy Actions and Statements," *International Journal of Central Banking*, May 2005, 55–93.

Gürkaynak, R. S., B. Sack, and J. H. Wright (2007): "The U.S. Treasury Yield Curve: 1961 to the Present," *Journal of Monetary Economics*, 54, 2291–2304.

Gürkaynak, R. S. and J. H. Wright (2012): "Macroeconomics and the Term Structure," *Journal of Economic Literature*, 50, 331–367.

Hayashi, F. and J. Koeda (2019): "Exiting from QE," *Quantitative Economics*, 10, 1069–1107.

Ichiue, H. and Y. Ueno (2015): "Monetary policy and the yield curve at zero interest," *Journal of The Japanese and International Economies*, 38, 1371–1376.

Joslin, S., M. Priebsch, and K. J. Singleton (2014): "Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risks," *Journal of Finance*, 69, 1197–1233.

Joslin, S., K. J. Singleton, and H. Zhu (2011): "A New Perspective on Gaussian Dynamic Term Structure Models," *Review of Financial Studies*, 24, 926–970.

Jung, T., Y. Teranishi, and T. Watanabe (2005): "Optimal Monetary Policy at the Zero-Interest-Rate Bound," *Journal of Money, Credit, and Banking*, 37, 1–23.

Kim, D. H. and K. J. Singleton (2012): "Term structure models and the zero bound: An empirical investigation of Japanese yields," *Journal of Econometrics*, 170, 32–49.

Koeda, J. and B. Wei (2023): "Quantifying Forward Guidance and Yield Curve Control," Working Paper.

Krippner, L. (2013a): "Measuring the Stance of Monetary Policy in Zero Lower Bound Environments," *Economics Letters*, 118, 135–138.

——— (2013b): "A tractable framework for zero-lower-bound Gaussian term structure models," CAMA Working Paper 49/2013.

——— (2020): "A Note of Caution on Shadow Rate Estimates," *Journal of Money, Credit and Banking*, 52, 951–962.

Krugman, P. R. (1998): "It's Baaack! Japan's Slump and the Return of the Liquidity Trap," *Brookings Papers on Economic Activity, Economic Studies Program, The Brookings Institution*, 29, 137–187.

McCracken, M. W. and S. Ng (2020): "FRED-QD: A Quarterly Database for Macroeconomic Research," St. Louis Fed Working Paper 2020-005B.

McKay, A., E. Nakamura, and J. Steinsson (2016): "The Power of Forward Guidance Revisited," *American Economic Review*, 106, 3133–3358.

Nakata, T., R. Ogaki, S. Schmidt, and P. Yoo (2019): "Attenuating the forward guidance puzzle: Implications for optimal monetary policy," *Journal of Economic Dynamics and Control*, 105, 90–106.

Piazzesi, M. (2005): "Bond Yields and the Federal Reserve," *Journal of Political Economy*, 113, 311–344.

Priebsch, M. A. (2013): "Computing Arbitrage-Free Yields in Multi-Factor Gaussian Shadow-Rate Term Structure Models," Finance and Economics Discussion Series, Federal Reserve Board.

Rosenbaum, S. (1961): "Moments of a Truncated Bivariate Normal Distribution," *Journal of the Royal Statistical Society, Series B (Methodological)*, 23, 405–408.

Rudebusch, G. D. and T. Wu (2008): "A Macro-Finance Model of the Term Structure, Monetary Policy and the Economy," *Economic Journal*, 118, 906–926.

Swanson, E. T. (2021): "Measuring the effects of federal reserve forward guidance and asset purchases on financial markets," *Journal of Monetary Economics*, 118, 32–53.

Ueda, K. (2012): "The effectiveness of non-traditional monetary policy measures: the case of the Bank of Japan," *The Japanese Economic Review*, 63, 1–22.

Ueno, Y., N. Baba, and Y. Sakurai (2006): "The Use of the Black Model of Interest Rates as Options for Monitoring the JGB Market Expectations," Working Paper, Bank of Japan.

Wright, J. H. (2011): "Term Premia and Inflation Uncertainty: Empirical Evidence from an International Panel Dataset," *American Economic Review*, 101, 1514–1534.

Wu, J. C. and F. D. Xia (2016): "Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound," *Journal of Money, Credit and Banking*, 48, 253–291.

Figures & Tables

Table 2: SRTSM Estimated Parameters

$400\mu_X$	-2.250	-0.099	-0.930
	(6.616)	(11.586)	(1.118)
ρ_{XX}	0.925	0.048	-0.371
	(0.181)	(0.151)	(2.142)
	0.026	0.875	1.283
	(0.314)	(0.200)	(1.316)
	-0.032	-0.011	0.915
	(0.014)	(0.017)	(0.479)
$\rho^{\mathbb{Q}}_{XX}$	0.995		
	(0.001)		
		0.893	1
		(0.008)	
			0.893
			(0.008)
$400r^{\mathbb{Q}}_{\infty}$	11.984		
	(2.665)		
$\sqrt{400}\Sigma_{XX}$	0.006		
	(0.023)		
	-0.005	0.008	
	(0.074)	(0.040)	
	-0.001	-0.001	0.002
	(0.024)	(0.015)	(0.003)
$\sigma_e \cdot 10^4$	7.801		
	(0.383)		
LLV	9614.4		

Note: This table reports parameter estimates for SRTSM. Standard errors, computed numerically based on the Hessian matrices, are reported in parentheses.

Table 3: MFGTSM Estimated Parameters

$400\mu_Z$	1.549	−10.174	0.125	−1.265	0.510
	(3.041)	(5.784)	(1.120)	(7.449)	(0.689)
ρ_Z	1.030	0.078	0.113	−0.105	0.105
	(0.075)	(0.069)	(0.445)	(0.053)	(0.219)
	−0.245	0.757	−0.133	0.206	0.206
	(0.142)	(0.118)	(0.586)	(0.088)	(0.347)
	−0.003	0.007	0.959	0.000	−0.014
	(0.024)	(0.028)	(0.148)	(0.017)	(0.082)
	−0.023	−0.166	−0.258	0.957	−0.102
	(0.121)	(0.120)	(1.262)	(0.244)	(0.254)
	0.015	0.011	0.000	0.039	0.939
	(0.019)	(0.016)	(0.065)	(0.027)	(0.042)
$\rho_{XX}^{\mathbb{Q}}$	0.999				
	(0.001)				
		0.884	1		
		(0.009)			
			0.884		
			(0.009)		
$400r_\infty^{\mathbb{Q}}$	9.359				
	(1.201)				
$\sqrt{400}\Sigma_Z$	0.010				
	(0.023)				
	−0.015	0.007			
	(0.052)	(0.021)			
	−0.003	0.000	0.002		
	(0.018)	(0.005)	(0.007)		
	0.011	0.001	0.006	0.012	
	(0.061)	(0.005)	(0.035)	(0.064)	
	0.000	0.000	0.001	0.001	0.003
	(0.008)	(0.002)	(0.008)	(0.009)	(0.006)
$\sigma_e \cdot 10^4$	8.750				
	(0.448)				
LLV	10308.0				

Note: This table reports parameter estimates for MFGTSM. Standard errors, computed numerically based on the Hessian matrices, are reported in parentheses.

Table 4: MFSRTSM0 Estimated Parameters

$400\mu_Z$	1.691	−9.633	0.094	−1.004	0.510
	(2.831)	(4.250)	(0.516)	(12.131)	(1.203)
ρ_Z	1.012	0.071	0.114	−0.104	0.043
	(0.052)	(0.049)	(1.167)	(0.050)	(0.139)
	−0.159	0.845	−0.212	0.253	0.242
	(0.061)	(0.057)	(0.920)	(0.098)	(0.200)
	−0.005	0.007	0.882	0.000	−0.023
	(0.011)	(0.012)	(0.208)	(0.012)	(0.030)
	−0.070	−0.135	−0.489	0.945	−0.117
	(0.194)	(0.172)	(3.320)	(0.235)	(0.941)
	0.009	0.012	0.000	0.046	0.944
	(0.040)	(0.041)	(0.400)	(0.031)	(0.093)
$\rho^{\mathbb{Q}}_{XX}$	0.995				
	(0.001)				
		0.896	1		
		(0.010)			
			0.896		
			(0.010)		
$400r^{\mathbb{Q}}_{\infty}$	11.038				
	(2.518)				
$\sqrt{400}\Sigma_Z$	0.005				
	(0.012)				
	−0.005	0.006			
	(0.027)	(0.019)			
	0.000	0.000	0.001		
	(0.005)	(0.006)	(0.007)		
	0.000	0.001	0.003	0.017	
	(1.207)	(0.416)	(0.466)	(0.140)	
	0.002	0.001	−0.001	0.002	0.002
	(0.596)	(0.171)	(0.281)	(0.183)	(0.757)
$\sigma_e \cdot 10^4$	8.012				
	(0.378)				
LLV	10543.7				

Note: This table reports parameter estimates for MFSRTSM0. Standard errors, computed numerically based on the Hessian matrices, are reported in parentheses.

Table 5: MFSRTSM Estimated Parameters

$400\mu_Z$	1.050	−10.731	−0.063	−0.244	0.130
	(0.369)	(1.120)	(0.003)	(0.124)	(0.014)
ρ_Z	1.002	0.057	0.164	−0.122	0.025
	(0.001)	(0.022)	(0.234)	(0.037)	(0.047)
	−0.178	0.796	−0.142	0.258	0.243
	(0.033)	(0.044)	(0.422)	(0.059)	(0.083)
	−0.009	0.005	0.923	0.000	−0.014
	(0.000)	(0.001)	(0.000)	(0.003)	(0.001)
	−0.067	0.004	−0.461	0.924	−0.121
	(0.005)	(0.018)	(0.015)	(0.001)	(0.001)
	0.006	0.019	0.000	0.020	1.011
	(0.003)	(0.009)	(0.014)	(0.006)	(0.000)
$\rho^{\mathbb{Q}}_{XX}$	0.995				
	(0.001)				
		0.893	1		
		(0.007)			
			0.893		
			(0.007)		
$400r^{\mathbb{Q}}_{\infty}$	11.606				
	(1.390)				
$\sqrt{400}\Sigma_Z$	0.005				
	(0.011)				
	−0.004	0.005			
	(0.012)	(0.014)			
	−0.001	−0.001	0.002		
	(0.013)	(0.017)	(0.006)		
	−0.002	0.005	0.005	0.016	
	(0.122)	(0.095)	(0.246)	(0.150)	
	0.000	0.003	−0.001	0.001	0.002
	(0.031)	(0.013)	(0.064)	(0.025)	(0.027)
$\sigma_e \cdot 10^4$	7.546				
	(0.386)				
LLV	10592.9				

Note: This table reports parameter estimates for MFSRTSM in which we set $\delta_2 = [0.5; 0.5]$ and $\underline{m} = 2\%$. Standard errors, computed numerically based on the Hessian matrices, are reported in parentheses.

Table 6: MFSRTSM Estimated Parameters including $\underline{m}$

$400\mu_Z$	1.591	−9.140	−0.958	−0.039	0.914
	(0.121)	(0.105)	(0.024)	(0.214)	(0.214)
ρ_Z	1.026	0.091	−0.026	−0.070	0.014
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	−0.149	0.763	0.183	0.186	0.232
	(0.000)	(0.000)	(0.000)	(0.010)	(0.000)
	−0.032	0.002	0.866	0.002	−0.001
	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)
	−0.086	−0.124	−0.329	0.859	−0.160
	(0.000)	(0.000)	(0.000)	(0.041)	(0.000)
	0.000	0.023	−0.014	−0.001	0.946
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$\rho^{\mathbb{Q}}_{XX}$	0.994				
	(0.001)				
		0.891	1		
		(0.001)			
			0.891		
			(0.001)		
$400r^{\mathbb{Q}}_{\infty}$	11.431				
	(0.030)				
$\sqrt{400}\Sigma_Z$	0.005				
	(0.021)				
	−0.004	0.005			
	(0.011)	(0.000)			
	−0.001	−0.001	0.002		
	(0.021)	(0.000)	(0.000)		
	−0.003	0.016	0.003	0.001	
	(0.000)	(0.000)	(0.000)	(1.175)	
	−0.003	0.001	0.001	0.001	0.001
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$\sigma_e \cdot 10^4$	7.463		$\underline{\mathrm{m}}$	1.948	
	(0.000)			(0.033)	
LLV	10614.9				

Note: This table reports parameter estimates for MFSRTSM in which we set $\delta_2 = [0.5; 0.5]$, but estimate $\underline{m}$. Standard errors, computed numerically based on the Hessian matrices, are reported in parentheses.

Appendices

A Examples of Outcome-based Forward Guidance in Policy Statements

Table A-1: Examples of FOMC Statements by the Federal Reserve, 2009-2020

Date	Quotes
2009.3.18,	"The Committee ... continues to anticipate that economic conditions, including low rates of resource utilization, subdued inflation trends, and stable inflation expectations, are likely to warrant exceptionally low levels for the federal funds rate for an **extended period**."
2011.6.22	"the Committee ... currently anticipates that economic conditions–including low rates of resource utilization and a subdued outlook for inflation over the medium run–are likely to warrant exceptionally low levels for the federal funds rate for an **extended period**."
2011.8.9	"the Committee ... currently anticipates that economic conditions–including low rates of resource utilization and a subdued outlook for inflation over the medium run–are likely to warrant exceptionally low levels for the federal funds rate at least through mid-2013."
2012.1.25	"the Committee ... currently anticipates that economic conditions–including low rates of resource utilization and a subdued outlook for inflation over the medium run–are likely to warrant exceptionally low levels for the federal funds rate at least through late 2014."
2012.9.13	"the Committee ... currently anticipates that exceptionally low levels for the federal funds rate are likely to be warranted at least through mid-2015."
2012.12.12 through 2014.1.29	"the Committee ... currently anticipates that this exceptionally low range for the federal funds rate will be appropriate at least as long as the unemployment rate remains above 6-1/2 percent, inflation between one and two years ahead is projected to be no more than a half percentage point above the Committee's 2 percent longer-run goal"
2014.3.19	"With the unemployment rate nearing 6-1/2 percent, the Committee has updated its forward guidance. "
2020.9.17	The Committee expects to maintain an accommodative stance of monetary policy until these outcomes are achieved.

Note: This table lists some excerpts from FOMC statements by the Federal Reserve.

Table A-2: Policy Statements by other Major Central Banks

date	quotes
A. Bank of Japan	
1999.4.13	"(The BOJ will) continue to supply ample funds until the deflationary concern is dispelled." (A remark by governor Hayami in a Q& A session with the press. Translation by authors.)
1999.9.21	"The Bank of Japan has been pursuing an unprecedented accommodative monetary policy and is explicitly committed to continue this policy until deflationary concerns subside."
2000.8.11	"... the downward pressure on prices ... has markedly receded...deflationary concern has been dispelled, the condition for lifting the zero interest rate policy."
2003.10.10	"The BOJ is currently committed to maintaining the quantitative easing policy until the consumer price index (excluding fresh food, on a nationwide basis) registers stably a zero percent or an cincrease year on year."
2012.2.14	"The Bank will continue pursuing the powerful easing until it judges that the 1% goal is in sight."
2016.1.29	"The Bank will continue with "QQE with a Negative Interest Rate," aiming to achieve the price stability target of 2%, as long as it is necessary for maintaining that target in a stable manner."
B. Bank of Canada	
2009.6.4	"Conditional on the current outlook for inflation, the target overnight rate can be expected to remain at its current level until the end of the second quarter of 2010 in order to achieve the inflation target."
C. Swedish Riksbank	
2009.6.4	"The future direction for monetary policy will depend on how new information on economic developments abroad and in Sweden will affect the prospects for inflation and economic activity in Sweden."

Note: This table lists some excerpts from policy statements by the Bank of Japan, the Bank of Canada, and the Swedish Riksbank.

B Analytical Approximation to Forward Rates and Forward Term Premiums

We first list a lemma below summarizing well-known results about the moments of truncated bivariate normal distributions, which will be used in our derivation.

Lemma 5. *Suppose (x_1, x_2) follows a standard bivariate normal distribution with correlation ϱ; that is,*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \varrho \\ \varrho & 1 \end{bmatrix} \right),$$

then the following results hold:

$$\Pr\left[x_1 \geq \alpha_1, x_2 \geq \alpha_2\right] = F\left(-\alpha_1, -\alpha_2; \varrho\right),$$

and

$$\Pr\left[x_1 \geq \alpha_1, x_2 \geq \alpha_2\right] E\left[x_1 \left| x_1 \geq \alpha_1, x_2 \geq \alpha_2\right.\right] = h\left(\alpha_1, \alpha_2; \varrho\right) + \varrho h\left(\alpha_2, \alpha_1; \varrho\right),$$

and

$$\begin{aligned} & \Pr\left[x_1 \geq \alpha_1, x_2 \geq \alpha_2\right] E\left[x_1 x_2 \left| x_1 \geq \alpha_1, x_2 \geq \alpha_2\right.\right] \\ = \; & \varrho\left(\alpha_1 h\left(\alpha_1, \alpha_2; \varrho\right) + \alpha_2 h\left(\alpha_2, \alpha_1; \varrho\right) + F\left(-\alpha_1, -\alpha_2; \varrho\right)\right) + \left(1 - \varrho^2\right) f\left(\alpha_1, \alpha_2; \varrho\right), \end{aligned}$$

and

$$\begin{aligned} & \Pr\left[x_1 \geq \alpha_1, x_2 \geq \alpha_2\right] E\left[x_1^2 \left| x_1 \geq \alpha_1, x_2 \geq \alpha_2\right.\right] \\ = \; & F\left(-\alpha_1, -\alpha_2; \varrho\right) + \alpha_1 h\left(\alpha_1, \alpha_2; \varrho\right) + \varrho^2 \alpha_2 h\left(\alpha_2, a_1; \varrho\right) + \varrho\left(1 - \varrho^2\right) f\left(\alpha_1, \alpha_2; \varrho\right), \end{aligned}$$

where

$$\begin{aligned} f\left(x_1, x_2; \varrho\right) &= \frac{1}{\sqrt{1-\varrho^2}} \frac{1}{2\pi} \exp\left\{ -\frac{1}{2} \frac{x_1^2 - 2\varrho x_1 x_2 + x_2^2}{\left(1-\varrho^2\right)} \right\}, \\ F\left(\alpha_1, \alpha_2; \varrho\right) &= \int_{-\infty}^{\alpha_2} \int_{-\infty}^{\alpha_1} f\left(x_1, x_2; \varrho\right) dx_1 dx_2, \\ h\left(\alpha_1, \alpha_2; \varrho\right) &= \phi\left(\alpha_1\right) \Phi\left(\frac{\varrho \alpha_1 - \alpha_2}{\sqrt{1-\varrho^2}} \right). \end{aligned}$$

Proof. See Rosenbaum (1961). □

Proof of Proposition 1. For the sake of exposition, we assume $\underline{r} = 0$. The proof for a nonzero $\underline{r}$ is similar. The proof is largely the same as Krippner (2013), even for our extended model "MFSRTSM0." Therefore, we only provide a sketch of the key steps.

By a similar argument in Krippner (2013), we can approximate the instantaneous forward rate as

$$f_{t,\tau}^{MFSRTSM} \equiv f_{t,\tau}^{GTSM} + \mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[\max\left(-s_{t+\tau}, 0\right)\right],$$

where $\mathbb{E}_t^{\mathbb{Q}^{t+\tau}}[\cdot]$ denotes the expectation taken under the $(t+\tau)$-forward measure $\mathbb{Q}^{t+\tau}$.

Under the $(t+\tau)$-forward measure $\mathbb{Q}^{t+\tau}$,

$$s_{t+\tau} \sim N\left(f_{t,\tau}^{GTSM}, \left(\sigma_\tau^{\mathbb{Q}}\right)^2\right) \text{ under measure } \mathbb{Q}^{t+\tau},$$

thus

$$\begin{aligned}
& \mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[\max\left(-s_{t+\tau}, 0\right)\right] \\
= \; & -\int_{-\infty}^{0} s_{t+\tau} \frac{1}{\sqrt{2\pi}\sigma_\tau^{\mathbb{Q}}} \exp\left[-\frac{\left(s_{t+\tau} - f_{t,\tau}^{GTSM}\right)^2}{2\left(\sigma_\tau^{\mathbb{Q}}\right)^2}\right] ds_{t+\tau} \\
= \; & -f_{t,\tau}^{GTSM}\Phi\left(\frac{-f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}}\right) - \frac{\sigma_\tau^{\mathbb{Q}}}{\sqrt{2\pi}} \int_{-\infty}^{-f_{t,\tau}^{GTSM}/\sigma_\tau^{\mathbb{Q}}} y \exp\left[-\frac{y^2}{2}\right] dy \\
= \; & -f_{t,\tau}^{GTSM}\Phi\left(-\frac{f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}}\right) + \sigma_\tau^{\mathbb{Q}}\phi\left(-\frac{f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_{t,\tau}^{MFSRTSM0} \; & \equiv \; f_{t,\tau}^{GTSM} + \mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[\max\left(-s_{t+\tau}, 0\right)\right] \\
& = \; f_{t,\tau}^{GTSM} - f_{t,\tau}^{GTSM}\Phi\left(-\frac{f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}}\right) + \sigma_\tau^{\mathbb{Q}}\phi\left(-\frac{f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}}\right) \\
& = \; f_{t,\tau}^{GTSM}\Phi\left(\frac{f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}}\right) + \sigma_\tau^{\mathbb{Q}}\phi\left(\frac{f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}}\right) \\
& = \; \sigma_\tau^{\mathbb{Q}} g\left(-\frac{f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}}\right).
\end{aligned}$$

□

Proof of Proposition 2. For the sake of exposition, we assume $\underline{r} = 0$. The proof for a nonzero $\underline{r}$ is similar.

Let $P_{t,\tau}^{MFSRTSM}$ denote the price a maturity-τ bond in MFSRTSM, that is, $P_{t,\tau}^{MFSRTSM} = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^\tau r_{t+u} du\right)\right]$ and the short rate r_t is defined in equation (12). By a similar argument as in Krippner (2013), we can approximate the forward bond price $P_{t,\tau,\delta}^{MFSRTSM} = P_{t,\tau+\delta}^{MFSRTSM}/P_{t,\tau}^{MFSRTSM}$ as follows

$$\begin{aligned}
P_{t,\tau,\delta}^{MFSRTSM} \; & = \; \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau+\delta} r_{t+u} du\right)\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} r_{t+u} du\right)\right]} \\
& \approx \; \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} s_{t+u} du\right) \exp\left(-\int_\tau^{\tau+\delta} r_{t+u} du\right)\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} s_{t+u} du\right)\right]}.
\end{aligned}$$

That is, we replace the short rate r_{t+u} by the shadow rate s_{t+u} for $u \in [0, \tau]$.

Let $r(t+\tau,\delta)$ denote the short rate between $t+\tau$ and $t+\tau+\delta$ and define the shadow rate $r(t+\tau,\delta)$ similarly for the same time interval. Then when δ is sufficiently small, we have

$$\begin{aligned}
& \exp\left(-\int_{\tau}^{\tau+\delta} r_{t+u} du\right) \\
\approx\ & \exp\left(-r(t+\tau,\delta)\delta\right) \\
=\ & \exp\left[-\max\left\{0, s(t+\tau,\delta)\right\}\delta 1_{\{m_{t+\tau}\geq \underline{m}\}}\right] \\
=\ & \min\left\{1, \exp\left[-s(t+\tau,\delta)\delta 1_{\{m_{t+\tau}\geq \underline{m}\}}\right]\right\} \\
=\ & \min\left\{1, P_{t+\tau,\delta}^{GTSM}\right\} 1_{\{m_{t+\tau}\geq \underline{m}\}} + 1_{\{m_{t+\tau}<\underline{m}\}} \\
=\ & P_{t+\tau,\delta}^{GTSM} - \max\left\{P_{t+\tau,\delta}^{GTSM} - 1, 0\right\} + 1_{\{m_{t+\tau}<\underline{m}\}} \max\left\{0, 1 - P_{t+\tau,\delta}^{GTSM}\right\},
\end{aligned}$$

where $P_{t+\tau,\delta}^{GTSM} = \exp\left[-s(t+\tau,\delta)\delta\right]$ denotes the shadow bond price at time $t+\tau$ with maturity δ.

Therefore, the forward bond price $P_{t,\tau,\delta}^{MFSRTSM}$ can be approximated as

$$\begin{aligned}
P_{t,\tau,\delta}^{MFSRTSM} &= P_{t,\tau,\delta}^{(1)} - P_{t,\tau,\delta}^{(2)} + P_{t,\tau,\delta}^{(3)} \\
&= \frac{P_{t,\tau+\delta}^{GTSM}}{P_{t,\tau}^{GTSM}} - \frac{C(t,\tau,\delta)}{P_{t,\tau}^{GTSM}} + \frac{D(t,\tau,\delta)}{P_{t,\tau}^{GTSM}},
\end{aligned}$$

where

$$\begin{aligned}
P_{t,\tau,\delta}^{(1)} &\equiv \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} s_{t+u} du\right) P_{t+\tau,\delta}^{GTSM}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} s_{t+u} du\right)\right]} = \frac{P_{t,\tau+\delta}^{GTSM}}{P_{t,\tau}^{GTSM}}, \\
P_{t,\tau,\delta}^{(2)} &\equiv \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} s_{t+u} du\right) \max\left\{P_{t+\tau,\delta}^{GTSM} - 1, 0\right\}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} s_{t+u} du\right)\right]} \equiv \frac{C_{t,\tau,\delta}}{P_{t,\tau}^{GTSM}}, \\
P_{t,\tau,\delta}^{(3)} &\equiv \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} s_{t+u} du\right) 1_{\{m_{t+\tau}<\underline{m}\}} \max\left\{0, 1 - P_{t+\tau,\delta}^{GTSM}\right\}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} s_{t+u} du\right)\right]} \equiv \frac{D_{t,\tau,\delta}}{P_{t,\tau}^{GTSM}}.
\end{aligned}$$

The instantaneous forward rate $f_{t,\tau}^{MFSRTSM} = -\lim_{\delta\to 0} \frac{1}{\delta} \log P_{t,\tau,\delta}^{MFSRTSM}$ satisfies the following standard relationship

$$\begin{aligned}
f_{t,\tau}^{MFSRTSM} &= -\lim_{\delta\to 0} \frac{1}{\delta} \log P_{t,\tau,\delta}^{MFSRTSM} \\
&= \lim_{\delta\to 0} \left\{-\frac{1}{\delta} \log\left[\frac{P_{t,\tau+\delta}^{GTSM} - C_{t,\tau,\delta} + D_{t,\tau,\delta}}{P_{t,\tau}^{GTSM}}\right]\right\} \\
&= -\lim_{\delta\to 0} \frac{d}{d\delta} \log\left[\frac{P_{t,\tau+\delta}^{GTSM} - C_{t,\tau,\delta} + D_{t,\tau,\delta}}{P_{t,\tau}^{GTSM}}\right] \\
&= -\lim_{\delta\to 0} \frac{d}{d\delta} \log\left[P_{t,\tau+\delta}^{GTSM} - C_{t,\tau,\delta} + D_{t,\tau,\delta}\right] \\
&= -\lim_{\delta\to 0} \frac{\frac{d}{d\delta}\left[P_{t,\tau+\delta}^{GTSM} - C_{t,\tau,\delta} + D_{t,\tau,\delta}\right]}{P_{t,\tau}^{GTSM}}, \\
&= -\frac{1}{P_{t,\tau}^{GTSM}} \lim_{\delta\to 0} \frac{d}{d\delta}\left[P_{t,\tau+\delta}^{GTSM} - C_{t,\tau,\delta} + D_{t,\tau,\delta}\right],
\end{aligned}$$

where in deriving the second-to-last equality we have used the results $\lim_{\delta\to 0} P^{GTSM}_{t,\tau+\delta} = P^{GTSM}_{t,\tau}$ and $\lim_{\delta\to 0} C_{t,\tau,\delta} = \lim_{\delta\to 0} D_{t,\tau,\delta} = 0$.

Next, because

$$\begin{aligned}
& \frac{1}{P^{GTSM}_{t,\tau}} \lim_{\delta\to 0} \frac{d}{d\delta} P^{GTSM}_{t,\tau+\delta} \\
= & \lim_{\delta\to 0}\lim_{\xi\to 0} \frac{P^{GTSM}_{t,\tau+\delta+\xi} - P^{GTSM}_{t,\tau+\delta}}{\xi} = \lim_{\xi\to 0}\lim_{\delta\to 0} \frac{P^{GTSM}_{t,\tau+\delta+\xi} - P^{GTSM}_{t,\tau+\delta}}{\xi} \\
= & \lim_{\xi\to 0} \frac{P^{GTSM}_{t,\tau+\xi} - P^{GTSM}_{t,\tau}}{\xi} = \frac{d}{d\tau} P^{GTSM}_{t,\tau} \\
= & -f^{GTSM}_{t,\tau} P^{GTSM}_{t,\tau},
\end{aligned}$$

we can thus rewrite the instantaneous forward rate as

$$f^{MFSRTSM}_{t,\tau} \equiv f^{GTSM}_{t,\tau} + z^{(1)}_{t,\tau} + z^{(2)}_{t,\tau},$$

where

$$\begin{aligned}
z^{(1)}_{t,\tau} & \equiv -\frac{1}{P^{GTSM}_{t,\tau}} \lim_{\delta\to 0} \frac{d}{d\delta} C_{t,\tau,\delta}, \\
z^{(2)}_{t,\tau} & \equiv -\frac{1}{P^{GTSM}_{t,\tau}} \lim_{\delta\to 0} \frac{d}{d\delta} D_{t,\tau,\delta}.
\end{aligned}$$

By a similar argument as in Krippner (2013), we can show that

$$\begin{aligned}
z^{(1)}_{t,\tau} & = \mathbb{E}^{\mathbb{Q}^{t+\tau}}_t \left[\max\left(-s_{t+\tau}, 0\right)\right] \\
z^{(2)}_{t,\tau} & = -\mathbb{E}^{\mathbb{Q}^{t+\tau}}_t \left[\max\left(s_{t+\tau}, 0\right) 1_{\{m_{t+\tau} < \underline{m}\}}\right],
\end{aligned}$$

where $\mathbb{E}^{\mathbb{Q}^{t+\tau}}_t[\cdot]$ denotes the expectation taken under the $(t+\tau)$-forward measure $\mathbb{Q}^{t+\tau}$.

We now derive $z^{(2)}_{t,\tau}$. Note that from Lemma 5, we have

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}^{t+\tau}}_t \left[\max\left(s_{t+\tau}, 0\right) 1_{\{m_{t+\tau} < \underline{m}\}}\right] \\
= & \mathbb{E}^{\mathbb{Q}^{t+\tau}}_t \left[\max\left(s_{t+\tau}, 0\right) 1_{\{-m_{t+\tau} > -\underline{m}\}}\right] \\
= & \Pr\left(s_{t+\tau} \geq 0, -m_{t+\tau} > -\underline{m}\right) \mathbb{E}^{\mathbb{Q}^{t+\tau}}_t \left[s_{t+\tau} \middle| s_{t+\tau} \geq 0, -m_{t+\tau} > -\underline{m}\right] \\
= & \sigma^{\mathbb{Q}}_\tau \Pr\left(s_{t+\tau} \geq 0, -m_{t+\tau} > -\underline{m}\right) \mathbb{E}^{\mathbb{Q}^{t+\tau}}_t \left[\frac{s_{t+\tau} - f^{GTSM}_{t,\tau}}{\sigma^{\mathbb{Q}}_\tau} \middle| s_{t+\tau} \geq 0, -m_{t+\tau} > -\underline{m}\right] \\
& + f^{GTSM}_{t,\tau} \Pr\left(s_{t+\tau} \geq 0, -m_{t+\tau} > -\underline{m}\right) \\
= & \sigma^{\mathbb{Q}}_\tau \left[h\left(\alpha_{t,\tau}, -\beta_{t,\tau}; -\varrho^{\mathbb{Q}}_\tau\right) - \varrho^{\mathbb{Q}}_\tau h\left(-\beta_{t,\tau}, \alpha_{t,\tau}; -\varrho^{\mathbb{Q}}_\tau\right)\right] + f^{GTSM}_{t,\tau} F\left(-\alpha_{t,\tau}, \beta_{t,\tau}; -\varrho^{\mathbb{Q}}_\tau\right),
\end{aligned}$$

where the expressions of $\alpha_{t,\tau} = \frac{\underline{r} - \mathbb{E}^{\mathbb{Q}^{t+\tau}}_t[s_{t+\tau}]}{\sigma^{\mathbb{Q}}_\tau} = \frac{\underline{r} - f^{GTSM}_{t,\tau}}{\sigma^{\mathbb{Q}}_\tau}$ and $\beta_{t,\tau} = \frac{\underline{m} - \mathbb{E}^{\mathbb{Q}^{t+\tau}}_t[m_{t+\tau}]}{\eta^{\mathbb{Q}}_\tau}$ are given in Lemma 6.

Therefore, we can derive the following approximation for the instantaneous forward rate $f_{t,\tau}^{MFSRTSM}$ in MFSRTSM:

$$\begin{aligned}
& f_{t,\tau}^{MFSRTSM} \\
= \; & f_{t,\tau}^{GTSM} + z_{t,\tau}^{(1)} + z_{t,\tau}^{(2)} \\
= \; & f_{t,\tau}^{MFSRTSM0} + z_{t,\tau}^{(2)} \\
= \; & \sigma_\tau^{\mathbb{Q}} g\left(-\alpha_{t,\tau}\right) - \sigma_\tau^{\mathbb{Q}}\left[h\left(\alpha_{t,\tau}, -\beta_{t,\tau}; -\varrho_\tau^{\mathbb{Q}}\right) - \varrho_\tau^{\mathbb{Q}} h\left(-\beta_{t,\tau}, \alpha_{t,\tau}; -\varrho_\tau^{\mathbb{Q}}\right)\right] - f_{t,\tau}^{GTSM} F\left(-\alpha_{t,\tau}, \beta_{t,\tau}; -\varrho_\tau^{\mathbb{Q}}\right) \\
= \; & \sigma_\tau^{\mathbb{Q}}\left[-\alpha_{t,\tau}\Phi\left(-\alpha_{t,\tau}\right) + \phi\left(\alpha_{t,\tau}\right) - h\left(\alpha_{t,\tau}, -\beta_{t,\tau}; -\varrho_\tau^{\mathbb{Q}}\right) + \varrho_\tau^{\mathbb{Q}} h\left(-\beta_{t,\tau}, \alpha_{t,\tau}; -\varrho_\tau^{\mathbb{Q}}\right)\right. \\
& \left. + F\left(-\alpha_{t,\tau}, \beta_{t,\tau}; -\varrho_\tau^{\mathbb{Q}}\right)\alpha_{t,\tau}\right] \\
= \; & \sigma_\tau^{\mathbb{Q}}\left[h\left(\alpha_{t,\tau}, \beta_{t,\tau}; \varrho_\tau^{\mathbb{Q}}\right) + \varrho_\tau^{\mathbb{Q}} h\left(\beta_{t,\tau}, \alpha_{t,\tau}; \varrho_\tau^{\mathbb{Q}}\right) - F\left(-\alpha_{t,\tau}, -\beta_{t,\tau}; \varrho_\tau^{\mathbb{Q}}\right)\alpha_{t,\tau}\right] \\
\equiv \; & \sigma_\tau^{\mathbb{Q}} g\left(-\alpha_{t,\tau}, -\beta_{t,\tau}; \varrho_\tau^{\mathbb{Q}}\right)
\end{aligned}$$

where in deriving the second-to-last equality we have used the following results

$$\begin{aligned}
h\left(-\beta, \alpha; -\varrho\right) &= \phi\left(-\beta\right)\Phi\left(\frac{\rho\beta - \alpha}{\sqrt{1-\rho^2}}\right) = \phi\left(\beta\right)\Phi\left(\frac{\rho\beta - \alpha}{\sqrt{1-\rho^2}}\right) = h\left(\beta, \alpha; \varrho\right), \\
h\left(\alpha, -\beta; -\varrho\right) + h\left(\alpha, \beta; \varrho\right) &= \phi\left(\alpha\right)\Phi\left(-\frac{\rho\alpha - \beta}{\sqrt{1-\rho^2}}\right) + \phi\left(\alpha\right)\Phi\left(\frac{\rho\alpha - \beta}{\sqrt{1-\rho^2}}\right) = \phi\left(\alpha\right), \\
F\left(-\alpha, \beta; -\varrho\right) + F\left(-\alpha, -\beta; \varrho\right) &= \Phi\left(-\alpha\right).
\end{aligned}$$

□

Lemma 6. *Under the $(t+\tau)$-forward measure $\mathbb{Q}^{t+\tau}$, $s_{t+\tau}$ and $m_{t+\tau}$ has the following normal distribution:*

$$\begin{bmatrix} s_{t+\tau} \\ m_{t+\tau} \end{bmatrix} \sim N\left(\begin{bmatrix} f_{t,\tau}^{GTSM} \\ \theta_2^* \end{bmatrix}, \begin{bmatrix} \left(\sigma_\tau^{\mathbb{Q}}\right)^2 & \varrho_\tau^{\mathbb{Q}}\sigma_\tau^{\mathbb{Q}}\eta_\tau^{\mathbb{Q}} \\ \varrho_\tau^{\mathbb{Q}}\sigma_\tau^{\mathbb{Q}}\eta_\tau^{\mathbb{Q}} & \left(\eta_\tau^{\mathbb{Q}}\right)^2 \end{bmatrix}\right),$$

where $\sigma_\tau^{\mathbb{Q}}$, $\eta_\tau^{\mathbb{Q}}$, and $\varrho_\tau^{\mathbb{Q}}$ are given in the proof.

Proof of Lemma 6. Let $d\widetilde{W}_{1,s}$ denote the Wiener process under the measure $\mathbb{Q}^{t+\tau}$ for $s \in [0, t+\tau]$, then by Girsanov's Theorem (see Chapter 7 in Filipovic, 2009)

$$d\widetilde{W}_{1,s} = dW_{1,s}^* - v\left(s, t+\tau\right)' ds,$$

where $v\left(s, t+\tau\right) \equiv -\iota'\left(I - e^{-\kappa_{11}^*(t+\tau-s)}\right)\left(\kappa_{11}^*\right)^{-1}\sigma_{11}$. It follows that under the measure $\mathbb{Q}^{t+\tau}$, we have

$$dX_t = \sigma_{11} v\left(t, t+\tau\right)' dt - \kappa_{11}^* X_t dt + \sigma_{11} d\widetilde{W}_{1,t},$$

implying

$$\begin{aligned}
X_{t+\tau} &= e^{-\kappa_{11}^*\tau} X_t + \int_t^{t+\tau} e^{\kappa_{11}^*(s-t-\tau)}\left[\sigma_{11} v\left(s, t+\tau\right)' ds + \sigma_{11} d\widetilde{W}_{1,s}\right], \\
E_t^{\mathbb{Q}^{t+\tau}}\left[X_{t+\tau}\right] &= e^{-\kappa_{11}^*\tau} X_t - \int_t^{t+\tau} e^{\kappa_{11}^*(s-t-\tau)}\sigma_{11}\sigma_{11}'\left[I - e^{-\kappa_{11}^{*\prime}(t+\tau-s)}\right]\left(\kappa_{11}^{*\prime}\right)^{-1} ds\iota \\
&\equiv e^{-\kappa_{11}^*\tau} X_t - \widetilde{M}\iota,
\end{aligned}$$

where $\widetilde{M} \equiv \int_t^{t+\tau} e^{\kappa_{11}^*(s-t-\tau)}\sigma_{11}\sigma_{11}'\left[I - e^{-\kappa_{11}^{*\prime}(t+\tau-s)}\right](\kappa_{11}^{*\prime})^{-1} ds$.

First, we prove $E_t^{\mathbb{Q}^{t+\tau}}[s_{t+\tau}] = f_{t,\tau}^{GTSM}$. This is because

$$E_t^{\mathbb{Q}^{t+\tau}}[s_{t+\tau}] = \delta_0 + \iota' E_t^{\mathbb{Q}^{t+\tau}}[X_{t+\tau}] \equiv \delta_0 - \iota'\widetilde{M}\iota + b_\tau' X_t,$$

To prove $E_t^{\mathbb{Q}^{t+\tau}}[s_{t+\tau}] = f_{t,\tau}^{GTSM} = a_\tau + b_\tau' X_t$, we just need to prove that $a_\tau = \delta_0 - \iota'\widetilde{M}\iota$, or equivalently, $\iota'\widetilde{M}\iota = \frac{1}{2}\iota'\widetilde{N}\iota$, where $\widetilde{N} \equiv (\kappa_{11}^*)^{-1}(I - \exp(-\kappa_{11}^*\tau))\sigma_{11}\sigma_{11}'(I - \exp(-\kappa_{11}^{*\prime}\tau))(\kappa_{11}^{*\prime})^{-1}$ and $\iota'\widetilde{N}\iota = B_\tau'\sigma_{11}\sigma_{11}'B_\tau$. It is straightforward to show that $\widetilde{M} + \widetilde{M}' = \widetilde{N}$.[11] Therefore, $\iota'\widetilde{M}\iota = \iota'\widetilde{M}'\iota = \frac{1}{2}\iota'\left(\widetilde{M} + \widetilde{M}'\right)\iota = \frac{1}{2}\iota'\widetilde{N}\iota$.

Second, we derive the conditional expectation $E_t^{\mathbb{Q}^{t+\tau}}[m_{t+\tau}]$. Note that under the risk-neural measure $\mathbb{Q}$, the dynamics of the state variables is given by

$$\begin{aligned} dX_t &= -\kappa_{11}^* X_t dt + \sigma_{11} dW_{1,t}^*, \\ dM_t &= -\kappa_{21}^* X_t dt + \kappa_{22}^*(\theta_2^* - M_t)\,dt + \sigma_{21} dW_{1,t}^* + \sigma_{22} dW_{2,t}^*. \end{aligned}$$

Let $\widehat{M}_t = M_t - \chi X_t$ and choose the constant χ to be the solution to

$$\chi\kappa_{11}^* - \kappa_{22}^*\chi = \kappa_{21}^*. \tag{B-1}$$

Then

$$\begin{aligned} d\widehat{M}_t &= dM_t - \chi dX_t \\ &= -\kappa_{21}^* X_t dt + \kappa_{22}^*(\theta_2^* - M_t)\,dt + \sigma_{21} dW_{1,t}^* + \sigma_{22} dW_{2,t}^* \\ &\quad -\chi\left[-\kappa_{11}^* X_t dt + \sigma_{11} dW_{1,t}^*\right] \\ &= (-\kappa_{21}^* + \chi\kappa_{11}^*)\,X_t dt + \kappa_{22}^*\left(\theta_2^* - \left[\widehat{M}_t + \chi X_t\right]\right)dt \\ &\quad + (\sigma_{21} - \chi\sigma_{11})\,dW_{1,t}^* + \sigma_{22} dW_{2,t}^* \\ &= (-\kappa_{21}^* + \chi\kappa_{11}^* - \kappa_{22}^*\chi)\,X_t dt + \kappa_{22}^*\left(\theta_2^* - \widehat{M}_t\right)dt \\ &\quad + (\sigma_{21} - \chi\sigma_{11})\,dW_{1,t}^* + \sigma_{22} dW_{2,t}^* \\ &\equiv \kappa_{22}^*\left(\theta_2^* - \widehat{M}_t\right)dt + \widehat{\sigma}_{21} dW_{1,t}^* + \sigma_{22} dW_{2,t}^*, \end{aligned}$$

[11] This is because

$$\begin{aligned} \widetilde{M} &= \int_0^\tau e^{-\kappa_{11}^*(\tau-s)}\sigma_{11}\sigma_{11}'\left[I - e^{-\kappa_{11}^{*\prime}(\tau-s)}\right](\kappa_{11}^{*\prime})^{-1} ds = \int_0^\tau \int_s^\tau e^{-\kappa_{11}^*(\tau-s)}\sigma_{11}\sigma_{11}' e^{-\kappa_{11}^{*\prime}(\tau-u)} du ds, \\ \widetilde{M}' &= \int_0^\tau \int_s^\tau e^{-\kappa_{11}^*(\tau-u)}\sigma_{11}\sigma_{11}' e^{-\kappa_{11}^{*\prime}(\tau-s)} du ds = \int_0^\tau \int_0^u e^{-\kappa_{11}^*(\tau-u)}\sigma_{11}\sigma_{11}' e^{-\kappa_{11}^{*\prime}(\tau-s)} ds du \\ &= \int_0^\tau \int_0^s e^{-\kappa_{11}^*(\tau-s)}\sigma_{11}\sigma_{11}' e^{-\kappa_{11}^{*\prime}(\tau-u)} du ds, \end{aligned}$$

and

$$\widetilde{M} + \widetilde{M}' = \int_0^\tau \int_s^\tau e^{-\kappa_{11}^*(\tau-s)}\sigma_{11}\sigma_{11}' e^{-\kappa_{11}^{*\prime}(\tau-u)} du ds + \int_0^\tau \int_0^s e^{-\kappa_{11}^*(\tau-s)}\sigma_{11}\sigma_{11}' e^{-\kappa_{11}^{*\prime}(\tau-u)} du ds = \widetilde{N}.$$

where $\widehat{\sigma}_{21} \equiv \sigma_{21} - \chi\sigma_{11}$.

The dynamics of $\widehat{M}_t$ under the forward measure $\mathbb{Q}^{t+\tau}$ is thus given by

$$d\widehat{M}_t = \widehat{\sigma}_{21} v\left(t, t+\tau\right)' dt + \kappa_{22}^* \left(\theta_2^* - \widehat{M}_t\right) dt + \widehat{\sigma}_{21} d\widetilde{W}_{1,t} + \sigma_{22} dW_{2,t}^*,$$

implying

$$\begin{aligned} E_t^{\mathbb{Q}^{t+\tau}}\left[\widehat{M}_{t+\tau}\right] &= \theta_2^* + e^{-\kappa_{22}^*\tau}\left(\widehat{M}_t - \theta_2^*\right) + \int_t^{t+\tau} e^{\kappa_{22}^* s-(t+\tau)} \widehat{\sigma}_{21} v\left(s, t+\tau\right)' ds \\ &= \theta_2^* + e^{-\kappa_{22}^*\tau}\left(\widehat{M}_t - \theta_2^*\right) - \int_0^{\tau} e^{-\kappa_{22}^*(\tau-s)} \widehat{\sigma}_{21} \sigma_{11}' \left(\kappa_{11}^{*\prime}\right)^{-1} \left(I - e^{-\kappa_{11}^{*\prime}(\tau-s)}\right) \iota ds. \end{aligned}$$

Therefore,

$$\begin{aligned} E_t^{\mathbb{Q}^{t+\tau}}\left[m_{t+\tau}\right] &= E_t^{\mathbb{Q}^{t+\tau}}\left[\delta_2' M_{t+\tau}\right] = \delta_2' E_t^{\mathbb{Q}^{t+\tau}}\left[\widehat{M}_{t+\tau}\right] + \delta_2' \chi E_t^{\mathbb{Q}^{t+\tau}}\left[X_{t+\tau}\right] \\ &= \delta_2' \left[\begin{array}{c} \theta_2^* + e^{-\kappa_{22}^*\tau}\left(\widehat{M}_t - \theta_2^*\right) \\ -\int_0^{\tau} e^{-\kappa_{22}^*(\tau-s)} \widehat{\sigma}_{21} \sigma_{11}' \left(\kappa_{11}^{*\prime}\right)^{-1} \left(I - e^{-\kappa_{11}^{*\prime}(\tau-s)}\right) \iota ds \end{array}\right] + \delta_2' \chi \left[e^{-\kappa_{11}^*\tau} X_t - \widetilde{M}\iota\right] \\ &\equiv c_\tau + d_\tau' M_t + e_\tau' X_t, \end{aligned}$$

where[12]

$$c_\tau \equiv \delta_2' \left[\left(I - e^{-\kappa_{22}^*\tau}\right)\theta_2^* - \int_0^{\tau} e^{-\kappa_{22}^*(\tau-s)} \widehat{\sigma}_{21} \sigma_{11}' \kappa_{11}^{*\prime -1} \left(I - e^{-\kappa_{11}^{*\prime}(\tau-s)}\right) \iota ds - \chi \widetilde{M}\iota\right], \tag{B-2a}$$

$$d_\tau' \equiv \delta_2' e^{-\kappa_{22}^*\tau}, \tag{B-2b}$$

$$e_\tau' \equiv \delta_2' \chi e^{-\kappa_{11}^*\tau}. \tag{B-2c}$$

Next, we calculate the time-t conditional variances of $s_{t+\tau}$ and $m_{t+\tau}$ as well as their correlation under the forward measure $\mathbb{Q}^{t+\tau}$, which are the same as those under the risk-neutral measure $\mathbb{Q}$. Because

$$\begin{aligned} X_{t+\tau} &= e^{-\kappa_{11}^*\tau} X_t + \int_t^{t+\tau} e^{\kappa_{11}^*(s-t-\tau)} \sigma_{11} dW_{1,s}^*, \\ \widehat{M}_{t+\tau} &= e^{-\kappa_{22}^*\tau} \widehat{M}_t + \int_t^{t+\tau} e^{\kappa_{22}^*(s-t-\tau)} \widehat{\sigma}_{21} dW_{1,s}^* + \int_t^{t+\tau} e^{\kappa_{22}^*(s-t-\tau)} \sigma_{22} dW_{2,s}^*, \end{aligned}$$

we have

$$\begin{aligned} Var_t^{\mathbb{Q}}\left[X_{t+\tau}\right] &= \int_0^{\tau} e^{\kappa_{11}^*(s-\tau)} \sigma_{11} \sigma_{11}' e^{\kappa_{11}^{*\prime}(s-\tau)} ds, \\ Var_t^{\mathbb{Q}}\left[\widehat{M}_{t+\tau}\right] &= \int_0^{\tau} e^{\kappa_{22}^*(s-\tau)} \left(\widehat{\sigma}_{21}\widehat{\sigma}_{21}' + \sigma_{22}\sigma_{22}'\right) e^{\kappa_{22}^{*\prime}(s-\tau)} + ds, \\ Cov_t^{\mathbb{Q}}\left[X_{t+\tau}, \widehat{M}_{t+\tau}'\right] &= \int_0^{\tau} e^{\kappa_{11}^*(s-\tau)} \sigma_{11} \widehat{\sigma}_{21}' e^{\kappa_{22}^{*\prime}(s-\tau)} ds. \end{aligned}$$

[12] In the special case with only one macro factor (i.e., $n_M = 1$), then the integral in (B-2a) has a closed-form expression: $c_\tau = \left[\begin{array}{c} \left(1 - e^{-\kappa_{22}^*\tau}\right)\theta_2^* - \frac{1-e^{-\kappa_{22}^*\tau}}{\kappa_{22}^*} \widehat{\sigma}_{21}\sigma_{11}' \left(\kappa_{11}^{*\prime}\right)^{-1} \iota \\ +\widehat{\sigma}_{21}\sigma_{11}' \left(\kappa_{11}^{*\prime}\right)^{-1} \left(I - e^{-\left(\kappa_{11}^{*\prime} + \kappa_{22} I\right)\tau}\right) \left(\kappa_{11}^{*\prime} + \kappa_{22} I\right)^{-1} \iota - \chi \widetilde{M}\iota \end{array}\right]$.

Note that

$$Cov_t^{\mathbb{Q}}\left[X_{t+\tau}, M'_{t+\tau}\right] = Cov_t^{\mathbb{Q}}\left[X_{t+\tau}, \widehat{M}'_{t+\tau}\right] + Var_t^{\mathbb{Q}}\left[X_{t+\tau}\right]\chi',$$

$$Var_t^{\mathbb{Q}}\left[M_{t+\tau}\right] = Var_t^{\mathbb{Q}}\left[\widehat{M}_{t+\tau}\right] - \chi Var_t^{\mathbb{Q}}\left[X_{t+\tau}\right]\chi' + \chi Cov_t^{\mathbb{Q}}\left[X_{t+\tau}, M'_{t+\tau}\right] + Cov_t^{\mathbb{Q}}\left[M_{t+\tau}, X'_{t+\tau}\right]\chi.$$

Therefore,

$$\sigma_\tau^{\mathbb{Q}} \equiv \sqrt{Var_t^{\mathbb{Q}}\left[s_{t+\tau}\right]} = \iota' Var_t^{\mathbb{Q}}\left[X_{t+\tau}\right]\iota, \tag{B-3a}$$

$$\eta_\tau^{\mathbb{Q}} \equiv \sqrt{Var_t^{\mathbb{Q}}\left[m_{t+\tau}\right]} = \delta_2' Var_t^{\mathbb{Q}}\left[M_{t+\tau}\right]\delta_2, \tag{B-3b}$$

$$\varrho_\tau^{\mathbb{Q}} \equiv \frac{Cov_t^{\mathbb{Q}}\left[s_{t+\tau}, m_{t+\tau}\right]}{\sigma_\tau^{\mathbb{Q}}\eta_\tau^{\mathbb{Q}}} = \frac{\iota' Cov_t^{\mathbb{Q}}\left[X_{t+\tau}, M'_{t+\tau}\right]\delta_2}{\sigma_\tau^{\mathbb{Q}}\eta_\tau^{\mathbb{Q}}}. \tag{B-3c}$$

□

Proof of Corollary 3. Suppose $\kappa_{21}^* = \sigma_{21} = 0$ and thus the bond factors X_t and the macro factors M_t are *independent* of each other under the risk-neutral measure $\mathbb{Q}$. In fact, it follows that $\chi = 0$, $\widehat{\sigma}_{21} \equiv \sigma_{21} - \chi\sigma_{11}$, and $Cov_t^{\mathbb{Q}}\left[X_{t+\tau}, M'_{t+\tau}\right] = 0$. As before we assume $\underline{r} = 0$ in the proof for the sake of exposition.

In this special case,

$$\begin{aligned} z_{t,\tau}^{(2)} &= -\mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[\max\left(s_{t+\tau}, 0\right) 1_{\{m_{t+\tau}<\underline{m}\}}\right] = -\mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[\max\left(s_{t+\tau}, 0\right)\right]\mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[1_{\{m_{t+\tau}<\underline{m}\}}\right] \\ &= -\sigma_\tau^{\mathbb{Q}} g\left(-\alpha_{t,\tau}\right)\Phi\left(\beta_{t,\tau}\right), \end{aligned}$$

where in deriving the last equality we have used the following results

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[\max\left(s_{t+\tau}, 0\right)\right] \\ =\ & \Pr\left(s_{t+\tau} \geq 0\right)\mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[s_{t+\tau} \middle| s_{t+\tau} \geq 0\right] \\ =\ & \sigma_\tau^{\mathbb{Q}} \Pr\left(s_{t+\tau} \geq 0\right)\mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[\frac{s_{t+\tau} - f_{t,\tau}^{GTSM}}{\sigma_\tau^{\mathbb{Q}}} \middle| s_{t+\tau} \geq 0\right] + f_{t,\tau}^{GTSM}\Pr\left(s_{t+\tau} \geq 0\right) \\ =\ & \sigma_\tau^{\mathbb{Q}} g\left(-\alpha_{t,\tau}\right), \end{aligned}$$

and

$$\mathbb{E}_t^{\mathbb{Q}^{t+\tau}}\left[1_{\{m_{t+\tau}<\underline{m}\}}\right] = \Phi\left(\beta_{t,\tau}\right).$$

Therefore,

$$f_{t,\tau}^{MFSRTSM} = f_{t,\tau}^{MFSRTSM0} + z_{t,\tau}^{(2)} = \left(1 - \Phi\left(\beta_{t,\tau}\right)\right)\sigma_\tau^{\mathbb{Q}} g\left(-\alpha_{t,\tau}\right) = \Phi\left(-\beta_{t,\tau}\right)\sigma_\tau^{\mathbb{Q}} g\left(-\alpha_{t,\tau}\right).$$

□

C Forward-Rate Approximation under Discrete-Time MFSRTSM

In this appendix, we provide the proof of Proposition 4. In particular, we derive the approximation of the forward rate in equation (13).

Define

$$\begin{aligned}
\overline{a}_n &\equiv \delta_0 + \delta_1' \left(\sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \right) \mu_X^{\mathbb{Q}}, && \text{(C-1a)} \\
a_n &\equiv \overline{a}_n - \frac{1}{2} \delta_1' \left(\sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \right) \Sigma_{XX} \Sigma_{XX}' \left(\sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \right)' \delta_1, && \text{(C-1b)} \\
b_n' &\equiv \delta_1' \left(\rho_{XX}^{\mathbb{Q}} \right)^n, && \text{(C-1c)} \\
c_n &\equiv \delta_2' \left(\sum_{j=0}^{n-1} \left(\rho_{MM}^{\mathbb{Q}} \right)^j \right) \left(\mu_M^{\mathbb{Q}} - \chi \mu_X^{\mathbb{Q}} \right) + \delta_2' \chi \left(\sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \right) \mu_X^{\mathbb{Q}}, && \text{(C-1d)} \\
d_n' &\equiv \delta_2' \left(\rho_{MM}^{\mathbb{Q}} \right)^n, && \text{(C-1e)} \\
e_n' &\equiv -\delta_2' \left(\rho_{MM}^{\mathbb{Q}} \right)^n \chi + \delta_2' \chi \left(\rho_{XX}^{\mathbb{Q}} \right)^n, && \text{(C-1f)}
\end{aligned}$$

and

$$\begin{aligned}
\left(\sigma_n^{\mathbb{Q}} \right)^2 &\equiv \sum_{j=0}^{n-1} \delta_1' \left(\rho_{XX}^{\mathbb{Q}} \right)^j \Sigma_{XX} \Sigma_{XX}' \left(\left(\rho_{XX}^{\mathbb{Q}} \right)' \right)^j \delta_1, && \text{(C-1g)} \\
\left(\tau_n^{\mathbb{Q}} \right)^2 &\equiv \sum_{j=0}^{n-1} \delta_2' \chi \left(\rho_{XX}^{\mathbb{Q}} \right)^j \Sigma_{XX} \Sigma_{XX}' \left(\left(\rho_{XX}^{\mathbb{Q}} \right)' \right)^j \chi' \delta_2 && \text{(C-1h)} \\
&\quad + \sum_{j=0}^{n-1} \delta_2' \left(\rho_{MM}^{\mathbb{Q}} \right)^j \left(\left(\Sigma_{MX} - \chi \Sigma_{XX} \right) \left(\Sigma_{MX} - \chi \Sigma_{XX} \right)' + \Sigma_{MM} \Sigma_{MM}' \right) \left(\left(\rho_{MM}^{\mathbb{Q}} \right)' \right)^j \delta_2 \\
&\quad + \sum_{j=0}^{n-1} \delta_2' \chi \left(\rho_{XX}^{\mathbb{Q}} \right)^j \left(\Sigma_{XX} \left(\Sigma_{MX} - \chi \Sigma_{XX} \right)' \right) \left(\left(\rho_{MM}^{\mathbb{Q}} \right)' \right)^j \delta_2 \\
&\quad + \sum_{j=0}^{n-1} \delta_2' \left(\rho_{MM}^{\mathbb{Q}} \right)^j \left(\left(\Sigma_{MX} - \chi \Sigma_{XX} \right) \Sigma_{XX} \right) \left(\left(\rho_{XX}^{\mathbb{Q}} \right)' \right)^j \chi' \delta_2,
\end{aligned}$$

and

$$\varrho_n \equiv \frac{1}{\sigma_n^{\mathbb{Q}} \tau_n^{\mathbb{Q}}} \left[\begin{array}{c} \sum_{j=0}^{n-1} \delta_1 \left(\rho_{XX}^{\mathbb{Q}} \right)^j \left(\Sigma_{XX} \left(\Sigma_{MX} - \chi \Sigma_{XX} \right)' \right) \left(\left(\rho_{MM}^{\mathbb{Q}} \right)' \right)^j \delta_2 \\ + \sum_{j=0}^{n-1} \delta_1 \left(\sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \Sigma_{XX} \Sigma_{XX}' \left(\left(\rho_{XX}^{\mathbb{Q}} \right)' \right)^j \right) \chi' \delta_2 \end{array} \right], \qquad \text{(C-1i)}$$

where χ is a $n_M \times n_X$ matrix satisfying:

$$\rho_{MX}^{\mathbb{Q}} = \chi \rho_{XX}^{\mathbb{Q}} - \rho_{MM}^{\mathbb{Q}} \chi. \qquad \text{(C-2)}$$

Based on Lemma 5, we provide an analytical expression of $E_t^{\mathbb{Q}} \left[r_{t+n} \right]$ in the lemma below.

Lemma 7. *The following statement is true:*

$$E_t^{\mathbb{Q}} \left[r_{t+n} \right] = \underline{r} + \sigma_n^{\mathbb{Q}} g \left(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n \right),$$

where the function $g(\cdot)$ *is defined by*

$$g \left(-\alpha_{t,n}, -\beta_{t,n} \right) \equiv h \left(\alpha_{t,n}, \beta_{t,n}; \varrho_n \right) + \varrho_n h \left(\beta_{t,n}, \alpha_{t,n}; \varrho_n \right) - F \left(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n \right) \alpha_{t,n}, \qquad \text{(C-3)}$$

and functions $F(\cdot, \cdot; \varrho_n)$ *and* $h(\cdot, \cdot; \varrho_n)$ *are defined in Lemma 5.*

Proof. First, because the shadow rate is affine in the yield curve factors, it is thus conditionally normally distributed under the risk-neutral measure. As shown in Wu and Xia (2016), the conditional mean and variance of the shadow rate are given by

$$\begin{aligned} E_t^{\mathbb{Q}}[s_{t+n}] &= \delta_0 + \delta_1' E_t^{\mathbb{Q}}[X_{t+n}] = \bar{a}_n + b_n' X_t, \\ Var_t^{\mathbb{Q}}[s_{t+n}] &= \delta_1' Var_t^{\mathbb{Q}}[X_{t+n}]\,\delta_1 = \left(\sigma_n^{\mathbb{Q}}\right)^2, \end{aligned}$$

and

$$\frac{1}{2}\left(Var_t^{\mathbb{Q}}\left[\sum\nolimits_{j=1}^{n} s_{t+j}\right] - Var_t^{\mathbb{Q}}\left[\sum\nolimits_{j=1}^{n-1} s_{t+j}\right]\right) = \bar{a}_n - a_n.$$

Next, define

$$\widetilde{r}_{t+n} \equiv \frac{r_{t+n} - E_t^{\mathbb{Q}}[s_{t+n}]}{\sigma_n^{\mathbb{Q}}}, \widetilde{s}_{t+n} \equiv \frac{s_{t+n} - E_t^{\mathbb{Q}}[s_{t+n}]}{\sigma_n^{\mathbb{Q}}}, \widetilde{m}_{t+n} \equiv \frac{m_{t+n} - E_t^{\mathbb{Q}}[m_{t+n}]}{\tau_n^{\mathbb{Q}}}.$$

Then we have

$$E_t^{\mathbb{Q}}[r_{t+n}] = \sigma_n^{\mathbb{Q}} E_t^{\mathbb{Q}}[\widetilde{r}_{t+n}] + E_t^{\mathbb{Q}}[s_{t+n}],$$

and, as a result of the liftoff condition,

$$\begin{aligned} & E_t^{\mathbb{Q}}[\widetilde{r}_{t+n}] \\ = \; & \Pr[\widetilde{s}_{t+n} \geq \alpha_{t,n}, \widetilde{m}_{t+n} \geq \beta_{t,n}]\, E_t^{\mathbb{Q}}[\widetilde{s}_{t+n} \,|\widetilde{s}_{t+n} \geq \alpha_{t,n}, \widetilde{m}_{t+n} \geq \beta_{t,n}] \\ & + (1 - \Pr[\widetilde{s}_{t+n} \geq \alpha_{t,n}, \widetilde{m}_{t+n} \geq \beta_{t,n}])\,\alpha_{t,n} \\ = \; & h(\alpha_{t,n}, \beta_{t,n}; \varrho_n) + \varrho_n h(\beta_{t,n}, \alpha_{t,n}; \varrho_n) + (1 - F(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n))\,\alpha_{t,n}, \end{aligned}$$

where in deriving the last equality of the second equation we have used the results from Lemma 5, and we use ϱ_n to denote the correlation between $\widetilde{s}_{t+n}$ and $\widetilde{m}_{t+n}$ under the risk-neutral measure. Note that we will prove later that such correlation is indeed equal to ϱ_n given by equation (C-1i).

Lastly, we derive $E_t^{\mathbb{Q}}[r_{t+n}]$ analytically as follows:

$$\begin{aligned} E_t^{\mathbb{Q}}[r_{t+n}] &= \sigma_n^{\mathbb{Q}} E_t^{\mathbb{Q}}[\widetilde{r}_{t+n}] + E_t^{\mathbb{Q}}[s_{t+n}] \\ &= \sigma_n^{\mathbb{Q}} \left[\begin{array}{c} h(\alpha_{t,n}, \beta_{t,n}; \varrho_n) + \varrho_n h(\beta_{t,n}, \alpha_{t,n}; \varrho_n) \\ + (1 - F(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n))\,\alpha_{t,n} \end{array}\right] + \bar{a}_n + b_n' X_t \\ &= \underline{r} + \sigma_n^{\mathbb{Q}} [h(\alpha_{t,n}, \beta_{t,n}; \varrho_n) + \varrho_n h(\beta_{t,n}, \alpha_{t,n}; \varrho_n) - F(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n)\,\alpha_{t,n}] \\ &\equiv \underline{r} + \sigma_n^{\mathbb{Q}} g(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n), \end{aligned}$$

where in deriving the second-to-last equality we have used the definition $\alpha_{t,n} \equiv \left(\underline{r} - E_t^{\mathbb{Q}}[s_{t+n}]\right)/\sigma_n^{\mathbb{Q}}$. $\square$

Next, note that following Wu and Xia (2016), the forward rate can be approximated as:

$$f_{t,n,n+1} \approx E_t^{\mathbb{Q}}[r_{t+n}] - \frac{1}{2}\left(Var_t^{\mathbb{Q}}\left[\sum\nolimits_{j=1}^{n} r_{t+j}\right] - Var_t^{\mathbb{Q}}\left[\sum\nolimits_{j=1}^{n-1} r_{t+j}\right]\right), \tag{C-4}$$

where the second term can be further approximated as follows:

$$
\begin{aligned}
& \frac{1}{2}\left(Var_t^{\mathbb{Q}}\left[\sum\nolimits_{j=1}^{n} r_{t+j}\right] - Var_t^{\mathbb{Q}}\left[\sum\nolimits_{j=1}^{n-1} r_{t+j}\right]\right) \\
\approx\ & \Pr\left[\widetilde{s}_{t+n} \geq \alpha_{t,n}, \widetilde{m}_{t+n} \geq \beta_{t,n}\right] \times \frac{1}{2}\left(Var_t^{\mathbb{Q}}\left[\sum\nolimits_{j=1}^{n} s_{t+j}\right] - Var_t^{\mathbb{Q}}\left[\sum\nolimits_{j=1}^{n-1} s_{t+j}\right]\right), \\
=\ & F\left(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n\right)\left(\overline{a}_n - a_n\right).
\end{aligned}
$$

As a result, we have

$$
\begin{aligned}
f_{t,n,n+1} &\approx \underline{r} + \sigma_n^{\mathbb{Q}} g\left(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n\right) - F\left(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n\right)\left(\overline{a}_n - a_n\right) \\
&= \underline{r} + \sigma_n^{\mathbb{Q}} g\left(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n\right) - \sigma_n^{\mathbb{Q}} \frac{\partial g\left(-\alpha_{t,n}, -\beta_{t,n}; \varrho_n\right)}{\partial \alpha_{t,n}}\left(\overline{a}_n - a_n\right) \\
&\approx \underline{r} + \sigma_n^{\mathbb{Q}} g\left(\frac{a_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}, \frac{\overline{c}_n + d_n' M_t + e_n' X_t - \underline{m}}{\tau_n^{\mathbb{Q}}}; \varrho_n\right),
\end{aligned}
$$

where the first-order Taylor approximation is used in deriving the last equality, and the second-to-last equality holds because

$$
\begin{aligned}
\frac{\partial g\left(x_1, x_2; \varrho\right)}{\partial x_1} &= -\left[x_1 h\left(-x_1, -x_2; \varrho\right) + \frac{\varrho}{\sqrt{1-\varrho^2}}\phi\left(-x_1\right)\phi\left(\frac{-\varrho x_1 + x_2}{\sqrt{1-\varrho^2}}\right)\right] \\
&\quad -\varrho\left[-\frac{1}{\sqrt{1-\varrho^2}}\phi\left(-x_2\right)\phi\left(\frac{-\varrho x_2 + x_1}{\sqrt{1-\varrho^2}}\right)\right] + h\left(-x_1, -x_2; \varrho\right)x_1 + F\left(x_1, x_2; \varrho\right), \\
&= \frac{\varrho}{\sqrt{1-\varrho^2}}\left[\phi\left(x_2\right)\phi\left(\frac{x_1 - \varrho x_2}{\sqrt{1-\varrho^2}}\right) - \phi\left(x_1\right)\phi\left(\frac{\varrho x_1 - x_2}{\sqrt{1-\varrho^2}}\right)\right] + F\left(x_1, x_2; \varrho\right), \\
&= F\left(x_1, x_2; \varrho\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& \phi\left(x_2\right)\phi\left(\frac{x_1 - \varrho x_2}{\sqrt{1-\varrho^2}}\right) - \phi\left(x_1\right)\phi\left(\frac{\varrho x_1 - x_2}{\sqrt{1-\varrho^2}}\right) \\
=\ & \frac{1}{2\pi}\left[\exp\left(-\frac{x_2^2}{2} - \frac{\left(x_1 - \varrho x_2\right)^2}{2\left(1-\varrho^2\right)}\right) - \exp\left(-\frac{x_1^2}{2} - \frac{\left(\varrho x_1 - x_2\right)^2}{2\left(1-\varrho^2\right)}\right)\right], \\
=\ & \frac{1}{2\pi}\left[\exp\left(-\frac{\left(1-\varrho^2\right)x_2^2 + \left(x_1 - \varrho x_2\right)^2}{2\left(1-\varrho^2\right)}\right) - \exp\left(-\frac{x_1^2\left(1-\varrho^2\right) + \left(\varrho x_1 - x_2\right)^2}{2\left(1-\varrho^2\right)}\right)\right], \\
=\ & 0.
\end{aligned}
$$

Note that in a special case with $\varrho_n = 0$, $g\left(-\alpha_{t,n}, -\beta_{t,n}; 0\right) = \Phi\left(-\beta_{t,n}\right)\left[\phi\left(-\alpha_{t,n}\right) + (-\alpha_{t,n})\Phi\left(-\alpha_{t,n}\right)\right]$.

Thus, we have shown that indeed the forward rate can be approximated by equation (13). To complete the proof of Proposition 4, it is only necessary to prove that the correlation between $\widetilde{s}_{t+n}$ and $\widetilde{m}_{t+n}$ under the risk-neutral measure is indeed equal to ϱ_n, as given by equation (C-1i).

Note that such correlation is given by

$$
\begin{aligned}
Corr_t^{\mathbb{Q}}(\widetilde{s}_{t+n}, \widetilde{m}_{t+n}) &= \frac{Cov_t^{\mathbb{Q}}\left(\frac{s_{t+n}-E_t^{\mathbb{Q}}[s_{t+n}]}{\sigma_n^{\mathbb{Q}}}, \frac{m_{t+n}-E_t^{\mathbb{Q}}[m_{t+n}]}{\tau_n^{\mathbb{Q}}}\right)}{\sqrt{Var_t^{\mathbb{Q}}\left(\frac{s_{t+n}-E_t^{\mathbb{Q}}[s_{t+n}]}{\sigma_n^{\mathbb{Q}}}\right)}\sqrt{Var_t^{\mathbb{Q}}\left(\frac{m_{t+n}-E_t^{\mathbb{Q}}[m_{t+n}]}{\tau_n^{\mathbb{Q}}}\right)}} \\
&= \frac{Cov_t^{\mathbb{Q}}\left(s_{t+n}, m_{t+n}\right)}{\sqrt{Var_t^{\mathbb{Q}}\left(s_{t+n}\right)}\sqrt{Var_t^{\mathbb{Q}}\left(m_{t+n}\right)}} \\
&= Corr_t^{\mathbb{Q}}\left[s_{t+n}, m_{t+n}\right].
\end{aligned}
$$

We now derive $Var_t^{\mathbb{Q}}\left(s_{t+n}\right)$, $Var_t^{\mathbb{Q}}\left(m_{t+n}\right)$,and $Cov_t^{\mathbb{Q}}\left(s_{t+n}, m_{t+n}\right)$. Recall from Lemma 7 that $Var_t^{\mathbb{Q}}\left(s_{t+n}\right) = \sigma_n^{\mathbb{Q}}$. In the following lemma, we derive $Var_t^{\mathbb{Q}}\left(m_{t+n}\right)$ and $Cov_t^{\mathbb{Q}}\left(s_{t+n}, m_{t+n}\right)$. In particular, we show that $Var_t^{\mathbb{Q}}\left(m_{t+n}\right) = \left(\tau_n^{\mathbb{Q}}\right)^2$ and $Cov_t^{\mathbb{Q}}\left(s_{t+n}, m_{t+n}\right) = \varrho_n \sigma_n^{\mathbb{Q}} \tau_n^{\mathbb{Q}}$. That is, the conditional correlation under the risk-neutral measure between s_{t+n} and m_{t+n} is indeed equal to ϱ_n.

Lemma 8. *Given the expressions of c_n, d_n, e_n, and $\tau_n^{\mathbb{Q}}$, ϱ_n in equations (C-1d)-(C-1f), (C-1h)-(C-1i), the following results hold:*

$$
\begin{aligned}
E_t^{\mathbb{Q}}\left[m_{t+n}\right] &= c_n + d_n' M_t + e_n' X_t, \\
Var_t^{\mathbb{Q}}\left[m_{t+n}\right] &= \left(\tau_n^{\mathbb{Q}}\right)^2, \\
Cov_t^{\mathbb{Q}}(s_{t+n}, m_{t+n}) &= \varrho_n \sigma_n^{\mathbb{Q}} \tau_n^{\mathbb{Q}}.
\end{aligned}
$$

Proof. Recall that the factor dynamics under the $\mathbb{Q}$ measure is given by

$$
\begin{bmatrix} X_{t+1} \\ M_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_X^{\mathbb{Q}} \\ \mu_M^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} \rho_{XX}^{\mathbb{Q}} & 0 \\ \rho_{MX}^{\mathbb{Q}} & \rho_{MM}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} X_t \\ M_t \end{bmatrix} + \begin{bmatrix} \Sigma_{XX} & 0 \\ \Sigma_{MX} & \Sigma_{MM} \end{bmatrix} \begin{bmatrix} \epsilon_{X,t+1}^{\mathbb{Q}} \\ \epsilon_{M,t+1}^{\mathbb{Q}} \end{bmatrix}.
$$

Define:

$$
\widehat{M}_t = M_t - \chi X_t,
$$

where χ, defined in equation (C-2), is a $n_M \times n_X$ matrix satisfying $\rho_{MX}^{\mathbb{Q}} = \chi \rho_{XX}^{\mathbb{Q}} - \rho_{MM}^{\mathbb{Q}} \chi$. Therefore,

$$
\begin{bmatrix} X_t \\ \widehat{M}_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\chi & I \end{bmatrix} \begin{bmatrix} X_t \\ M_t \end{bmatrix},
$$

and

$$
\begin{bmatrix} X_t \\ M_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\chi & I \end{bmatrix}^{-1} \begin{bmatrix} X_t \\ \widehat{M}_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ \chi & I \end{bmatrix} \begin{bmatrix} X_t \\ \widehat{M}_t \end{bmatrix},
$$

implying:

$$\begin{aligned}
\begin{bmatrix} X_{t+1} \\ \widehat{M}_{t+1} \end{bmatrix} &= \begin{bmatrix} I & 0 \\ -\chi & I \end{bmatrix} \begin{bmatrix} \mu_X^{\mathbb{Q}} \\ \mu_M^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} I & 0 \\ -\chi & I \end{bmatrix} \begin{bmatrix} \rho_{XX}^{\mathbb{Q}} & 0 \\ \rho_{MX}^{\mathbb{Q}} & \rho_{MM}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} X_t \\ M_t \end{bmatrix} \\
&\quad + \begin{bmatrix} I & 0 \\ -\chi & I \end{bmatrix} \begin{bmatrix} \Sigma_{XX} & 0 \\ \Sigma_{MX} & \Sigma_{MM} \end{bmatrix} \begin{bmatrix} \epsilon_{X,t+1}^{\mathbb{Q}} \\ \epsilon_{M,t+1}^{\mathbb{Q}} \end{bmatrix}, \\
&= \begin{bmatrix} \mu_X^{\mathbb{Q}} \\ \mu_M^{\mathbb{Q}} - \chi \mu_X^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} I & 0 \\ -\chi & I \end{bmatrix} \begin{bmatrix} \rho_{XX}^{\mathbb{Q}} & 0 \\ \rho_{MX}^{\mathbb{Q}} & \rho_{MM}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} I & 0 \\ \chi & I \end{bmatrix} \begin{bmatrix} X_t \\ \widehat{M}_t \end{bmatrix} \\
&\quad + \begin{bmatrix} \Sigma_{XX} & 0 \\ \Sigma_{MX} - \chi \Sigma_{XX} & \Sigma_{MM} \end{bmatrix} \begin{bmatrix} \epsilon_{X,t+1}^{\mathbb{Q}} \\ \epsilon_{M,t+1}^{\mathbb{Q}} \end{bmatrix}, \\
&= \begin{bmatrix} \mu_X^{\mathbb{Q}} \\ \mu_M^{\mathbb{Q}} - \chi \mu_X^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} \rho_X^{\mathbb{Q}} & 0 \\ \rho_{MX}^{\mathbb{Q}} - \chi \rho_X^{\mathbb{Q}} + \rho_{MM}^{\mathbb{Q}} \chi & \rho_{MM}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} X_t \\ \widehat{M}_t \end{bmatrix} \\
&\quad + \begin{bmatrix} \Sigma_{XX} & 0 \\ \Sigma_{MX} - \chi \Sigma_{XX} & \Sigma_{MM} \end{bmatrix} \begin{bmatrix} \epsilon_{X,t+1}^{\mathbb{Q}} \\ \epsilon_{M,t+1}^{\mathbb{Q}} \end{bmatrix}, \\
&\equiv \begin{bmatrix} \mu_X^{\mathbb{Q}} \\ \widehat{\mu}_M^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} \rho_{XX}^{\mathbb{Q}} & 0 \\ 0 & \rho_{MM}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} X_t \\ \widehat{M}_t \end{bmatrix} + \begin{bmatrix} \Sigma_{XX} & 0 \\ \widehat{\Sigma}_{MX} & \Sigma_{MM} \end{bmatrix} \begin{bmatrix} \epsilon_{X,t+1}^{\mathbb{Q}} \\ \epsilon_{M,t+1}^{\mathbb{Q}} \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\mu}_M^{\mathbb{Q}} &\equiv \mu_M^{\mathbb{Q}} - \chi \mu_X^{\mathbb{Q}}, \\
\widehat{\Sigma}_{MX} &\equiv \Sigma_{MX} - \chi \Sigma_{XX}.
\end{aligned}$$

Because $\widehat{M}_{t+1} = \widehat{\mu}_M^{\mathbb{Q}} + \rho_{MM}^{\mathbb{Q}} \widehat{M}_t + \widehat{\Sigma}_{MX} \epsilon_{X,t+1}^{\mathbb{Q}} + \Sigma_{MM} \epsilon_{M,t+1}^{\mathbb{Q}}$, we have

$$\begin{aligned}
E_t^{\mathbb{Q}} \left[\widehat{M}_{t+n} \right] &= \sum_{j=0}^{n-1} \left(\rho_{MM}^{\mathbb{Q}} \right)^j \widehat{\mu}_M^{\mathbb{Q}} + \left(\rho_{MM}^{\mathbb{Q}} \right)^n \widehat{M}_t, \\
Var_t^{\mathbb{Q}} \left[\widehat{M}_{t+n} \right] &= \sum_{j=0}^{n-1} \left(\rho_{MM}^{\mathbb{Q}} \right)^j \left(\widehat{\Sigma}_{MX} \widehat{\Sigma}'_{MX} + \Sigma_{MM} \Sigma'_{MM} \right) \left(\left(\rho_{MM}^{\mathbb{Q}} \right)' \right)^j.
\end{aligned}$$

Furthermore, note that for $n \geq 1$,

$$Cov_t \left(X_{t+n}, \widehat{M}'_{t+n} \right) = \sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \left(\Sigma_{XX} \widehat{\Sigma}'_{MX} \right) \left(\left(\rho_{MM}^{\mathbb{Q}} \right)' \right)^j.$$

From the previous equation as well as the equation for $Var_t \left(X_{t+n} \right)$, we have

$$\begin{aligned}
& Cov_t \left(X_{t+n}, M'_{t+n} \right) \\
=\ & Cov_t \left(X_{t+n}, \widehat{M}'_{t+n} \right) + Var_t \left(X_{t+n} \right) \chi', \\
=\ & \sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \left(\Sigma_{XX} \widehat{\Sigma}'_{MX} \right) \left(\left(\rho_{MM}^{\mathbb{Q}} \right)' \right)^j + \left(\sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \Sigma_{XX} \Sigma'_{XX} \left(\left(\rho_{XX}^{\mathbb{Q}} \right)' \right)^j \right) \chi'.
\end{aligned}$$

Thus

$$\begin{aligned}
& Cov_t^{\mathbb{Q}} (s_{t+n}, m_{t+n}) \\
=\ & \delta_1 Cov_t^{\mathbb{Q}} (X_{t+n}, M_{t+n}) \delta_2, \\
=\ & \delta_1 \left[\sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \left(\Sigma_{XX} \widehat{\Sigma}'_{MX} \right) \left(\left(\rho_{MM}^{\mathbb{Q}} \right)' \right)^j + \chi \left(\sum_{j=0}^{n-1} \left(\rho_{XX}^{\mathbb{Q}} \right)^j \Sigma_{XX} \Sigma'_{XX} \left(\left(\rho_{XX}^{\mathbb{Q}} \right)' \right)^j \right) \right] \delta_2.
\end{aligned}$$

From $Var_t^{\mathbb{Q}}\left[\widehat{M}_{t+n}\right]=Var_t^{\mathbb{Q}}\left[M_{t+n}\right]+\chi Var_t^{\mathbb{Q}}\left[X_{t+n}\right]\chi'-\chi Cov_t^{\mathbb{Q}}\left(X_{t+n},M'_{t+n}\right)-Cov_t^{\mathbb{Q}}\left(M_{t+n},X'_{t+n}\right)\chi'$, we obtain

$$\begin{aligned}
& Var_t^{\mathbb{Q}}\left[M_{t+n}\right] \\
= \;& Var_t^{\mathbb{Q}}\left[\widehat{M}_{t+n}\right]-\chi Var_t^{\mathbb{Q}}\left[X_{t+n}\right]\chi'+\chi Cov_t\left(X_{t+n},M'_{t+n}\right)+Cov_t\left(M_{t+n},X'_{t+n}\right)\chi' \\
= \;& \sum_{j=0}^{n-1}\left(\rho_{MM}^{\mathbb{Q}}\right)^j\left(\widehat{\Sigma}_{MX}\widehat{\Sigma}'_{MX}+\Sigma_{MM}\Sigma'_{MM}\right)\left(\left(\rho_{MM}^{\mathbb{Q}}\right)'\right)^j \\
& -\chi\left(\sum_{j=0}^{n-1}\left(\rho_{XX}^{\mathbb{Q}}\right)^j\Sigma_{XX}\Sigma'_{XX}\left(\left(\rho_{XX}^{\mathbb{Q}}\right)'\right)^j\right)\chi' \\
& +\chi\left[\sum_{j=0}^{n-1}\left(\rho_{XX}^{\mathbb{Q}}\right)^j\left(\Sigma_{XX}\widehat{\Sigma}'_{MX}\right)\left(\left(\rho_{MM}^{\mathbb{Q}}\right)'\right)^j+\left(\sum_{j=0}^{n-1}\left(\rho_{XX}^{\mathbb{Q}}\right)^j\Sigma_{XX}\Sigma'_{XX}\left(\left(\rho_{XX}^{\mathbb{Q}}\right)'\right)^j\right)\chi'\right] \\
& +\left[\sum_{j=0}^{n-1}\left(\rho_{MM}^{\mathbb{Q}}\right)^j\left(\widehat{\Sigma}_{MX}\Sigma_{XX}\right)\left(\left(\rho_{XX}^{\mathbb{Q}}\right)'\right)^j+\chi\left(\sum_{j=0}^{n-1}\left(\rho_{XX}^{\mathbb{Q}}\right)^j\Sigma_{XX}\Sigma'_{XX}\left(\left(\rho_{XX}^{\mathbb{Q}}\right)'\right)^j\right)\right]\chi'.
\end{aligned}$$

That is,

$$\begin{aligned}
& Var_t^{\mathbb{Q}}\left[M_{t+n}\right] \\
= \;& \sum_{j=0}^{n-1}\left[\begin{array}{c}\left(\rho_{MM}^{\mathbb{Q}}\right)^j\left(\widehat{\Sigma}_{MX}\widehat{\Sigma}'_{MX}+\Sigma_{MM}\Sigma'_{MM}\right)\left(\left(\rho_{MM}^{\mathbb{Q}}\right)'\right)^j+\chi\left(\rho_{XX}^{\mathbb{Q}}\right)^j\Sigma_{XX}\Sigma'_{XX}\left(\left(\rho_{XX}^{\mathbb{Q}}\right)'\right)^j\chi' \\ +\chi\left(\rho_{XX}^{\mathbb{Q}}\right)^j\left(\Sigma_{XX}\widehat{\Sigma}'_{MX}\right)\left(\left(\rho_{MM}^{\mathbb{Q}}\right)'\right)^j+\left(\rho_{MM}^{\mathbb{Q}}\right)^j\left(\widehat{\Sigma}_{MX}\Sigma_{XX}\right)\left(\left(\rho_{XX}^{\mathbb{Q}}\right)'\right)^j\chi'\end{array}\right].
\end{aligned}$$

Therefore,

$$Var_t^{\mathbb{Q}}\left[m_{t+n}\right]=\delta'_2 Var_t^{\mathbb{Q}}\left[M_{t+n}\right]\delta_2=\left(\tau_n^{\mathbb{Q}}\right)^2,$$

and

$$\begin{aligned}
& Corr_t\left(s_{t+n},m_{t+n}\right) \\
= \;& \frac{\delta_1 Cov_t\left(X_{t+n},M'_{t+n}\right)\delta'_2}{\sigma_n^{\mathbb{Q}}\tau_n^{\mathbb{Q}}} \\
= \;& \frac{1}{\sigma_n^{\mathbb{Q}}\tau_n^{\mathbb{Q}}}\left[\begin{array}{c}\sum_{j=0}^{n-1}\delta_1\left(\rho_{XX}^{\mathbb{Q}}\right)^j\left(\Sigma_{XX}\widehat{\Sigma}'_{MX}\right)\left(\left(\rho_{MM}^{\mathbb{Q}}\right)'\right)^j\delta'_2 \\ +\sum_{j=0}^{n-1}\delta_1\left(\sum_{j=0}^{n-1}\left(\rho_{XX}^{\mathbb{Q}}\right)^j\Sigma_{XX}\Sigma'_{XX}\left(\left(\rho_{XX}^{\mathbb{Q}}\right)'\right)^j\right)\chi'\delta'_2\end{array}\right] \\
= \;& \varrho_n.
\end{aligned}$$

Lastly, we have

$$\begin{aligned}
E_t^{\mathbb{Q}}\left[M_{t+n}\right] &= E_t^{\mathbb{Q}}\left[\widehat{M}_{t+n}\right]+\chi E_t^{\mathbb{Q}}\left[X_{t+n}\right] \\
&= \left(\sum_{j=0}^{n-1}\left(\rho_{MM}^{\mathbb{Q}}\right)^j\widehat{\mu}_M^{\mathbb{Q}}+\left(\rho_{MM}^{\mathbb{Q}}\right)^n\widehat{M}_t\right)+\chi\left(\sum_{j=0}^{n-1}\left(\rho_{XX}^{\mathbb{Q}}\right)^j\mu_X^{\mathbb{Q}}+\left(\rho_{XX}^{\mathbb{Q}}\right)^n X_t\right)
\end{aligned}$$

and

$$
\begin{aligned}
E_t^{\mathbb{Q}}\left[m_{t+n}\right] &= E_t^{\mathbb{Q}}\left[\delta_2' M_{t+n}\right], \\
&= \delta_2'\left(\sum\nolimits_{j=0}^{n-1}\left(\rho_{MM}^{\mathbb{Q}}\right)^j \widehat{\mu}_M^{\mathbb{Q}} + \left(\rho_{MM}^{\mathbb{Q}}\right)^n\left(M_t - \chi X_t\right)\right) \\
&\quad + \delta_2'\chi\left(\sum\nolimits_{j=0}^{n-1}\left(\rho_{XX}^{\mathbb{Q}}\right)^j \mu_X^{\mathbb{Q}} + \left(\rho_{XX}^{\mathbb{Q}}\right)^n X_t\right), \\
&= \left(\delta_2'\sum\nolimits_{j=0}^{n-1}\left(\rho_{MM}^{\mathbb{Q}}\right)^j \widehat{\mu}_M^{\mathbb{Q}} + \delta_2'\chi\sum\nolimits_{j=0}^{n-1}\left(\rho_{XX}^{\mathbb{Q}}\right)^j \mu_X^{\mathbb{Q}}\right) \\
&\quad + \delta_2'\left(\rho_{MM}^{\mathbb{Q}}\right)^n M_t + \left(-\delta_2'\left(\rho_{MM}^{\mathbb{Q}}\right)^n\chi + \delta_2'\chi\left(\rho_{XX}^{\mathbb{Q}}\right)^n\right)X_t, \\
&\equiv c_n + d_n' M_t + e_n' X_t.
\end{aligned}
$$

□

D Extended Kalman Filter

D.1. State Space Representation under the MFSRTSM0

State-Space Representation. Suppose in each period t we observe one-period forward rates for totally J different maturities: $n_1, \cdots, n_J$. Let $F_t^o = \left(f_{t,n_1,n_1+1}, \cdots, f_{t,n_J,n_J+1}\right)'$ denote the vector of observed forward rates. We denote the vector of observables by $Y_t' = (F_t^{o\prime}, M_t')$.

We write the MFSRTSM0 as a non-linear state-space model. The measurement equations are given by

$$
Y_{t+1} = \begin{bmatrix} F_{t+1}^o \\ M_{t+1} \end{bmatrix} = \begin{bmatrix} G\left(X_{t+1}\right) + \eta_{t+1} \\ \mu_M + \rho_{MX}X_t + \rho_{MM}M_t + \Sigma_{MX}\epsilon_{X,t+1} + \Sigma_{MM}\epsilon_{M,t+1} \end{bmatrix},
$$

where $\eta_{t+1} \sim N\left(0, \omega I_{n_J}\right)$ and

$$
G\left(X_{t+1}\right) = \begin{bmatrix} \underline{r} + \sigma_{n_1}^{\mathbb{Q}} g\left(\frac{a_{n_1} + b_{n_1}' X_{t+1} - \underline{r}}{\sigma_{n_1}^{\mathbb{Q}}}\right) \\ \vdots \\ \underline{r} + \sigma_{n_J}^{\mathbb{Q}} g\left(\frac{a_{n_F} + b_{n_F}' X_{t+1} - \underline{r}}{\sigma_{n_J}^{\mathbb{Q}}}\right) \end{bmatrix}.
$$

The transition equation is given by

$$
\widehat{X}_{t+1} = \Phi_0 + \Phi_1\widehat{X}_t + \Phi_2 Y_t + v_t,
$$

where $v_t \sim N\left(0, \Sigma_v\right)$ and

$$
\widehat{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix}, \Phi_0 = \begin{bmatrix} \mu_X \\ 0 \end{bmatrix},\ \Phi_1 = \begin{bmatrix} \rho_{XX} & 0 \\ I & 0 \end{bmatrix},\ \Phi_2 = \begin{bmatrix} 0 & \rho_{XM} \\ 0 & 0 \end{bmatrix},\ \Sigma_v = \begin{bmatrix} \Sigma_{XX}\Sigma_{XX}' & \\ & 0 \end{bmatrix}.
$$

Extended Kalman Filter. Let $\widehat{X}_{t+1|t}$ denote time-t conditional expectation of $\widehat{X}_{t+1}$; that is,

$$
\widehat{X}_{t+1|t} = E\left[\begin{bmatrix} X_{t+1} \\ X_t \end{bmatrix} |\mathcal{F}_t\right] = \begin{bmatrix} E\left[X_{t+1}|\mathcal{F}_t\right] \\ E\left[X_t|\mathcal{F}_t\right] \end{bmatrix},
$$

where $\mathcal{F}_t$ denotes the information set at time t. $\widehat{X}_{t|t}$ is similarly defined. Let $\widehat{P}_{t+1|t} \equiv Var\left[\widehat{X}_{t+1}\left|\mathcal{F}_t\right.\right]$ and $\widehat{P}_{t|t} \equiv Var\left[\widehat{X}_t\left|\mathcal{F}_t\right.\right]$.

Starting with the initial value $\widehat{X}_{0|0}$ and $\widehat{P}_{0|0}$, we can implement the extended Kalman filter recursively as follows:

- Predict $\widehat{X}_{t+1|t}$ and $\widehat{P}_{t+1|t}$ as follows:

$$\begin{aligned}\widehat{X}_{t+1|t} &= \Phi_0 + \Phi_1 \widehat{X}_{t|t} + \Phi_2 Y_t, \\ \widehat{P}_{t+1|t} &= \Phi_1 \widehat{P}_{t|t} \Phi_1' + \Sigma_v.\end{aligned}$$

- Forecast Y_{t+1}:

$$\widehat{Y}_{t+1|t} = \begin{bmatrix} G\left(E\left[X_{t+1}\left|\mathcal{F}_t\right.\right]\right) \\ \mu_M + \rho_{MX} E\left[X_t\left|\mathcal{F}_t\right.\right] + \rho_{MM} M_t \end{bmatrix},$$

  and forecasting error

$$Y_{t+1} - \widehat{Y}_{t+1|t} = \begin{bmatrix} G'\left(E\left[X_{t+1}\left|\mathcal{F}_t\right.\right]\right)\left(X_{t+1} - E\left[X_{t+1}\left|\mathcal{F}_t\right.\right]\right) + \eta_{t+1} \\ \rho_{MX}\left(X_t - E\left[X_t\left|\mathcal{F}_t\right.\right]\right) + \Sigma_{MX}\epsilon_{X,t+1} + \Sigma_{MM}\epsilon_{M,t+1} \end{bmatrix},$$

  with forecast MSE given by

$$\begin{aligned}\widehat{V}_{t+1|t} &\equiv E\left[\left(Y_{t+1} - \widehat{Y}_{t+1|t}\right)\left(Y_{t+1} - \widehat{Y}_{t+1|t}\right)'\left|\mathcal{F}_t\right.\right] \\ &= \Psi_t \widehat{P}_{t+1|t} \Psi_t' + \begin{bmatrix} \omega I & \\ & \Sigma_{MX}\Sigma_{MX}' + \Sigma_{MM}\Sigma_{MM}' \end{bmatrix},\end{aligned}$$

  where

$$\Psi_t = \begin{bmatrix} G'\left(E\left[X_{t+1}\left|\mathcal{F}_t\right.\right]\right) & 0_{J\times n_X} \\ 0_{n_M\times n_X} & \rho_{MX} \end{bmatrix}.$$

- Update $\widehat{X}_{t+1|t+1}$ and $\widehat{P}_{t+1|t+1}$ as follows:

$$\begin{aligned}\widehat{X}_{t+1|t+1} &= \widehat{X}_{t+1|t} + K_{t+1}\left(F_{t+1}^o - G\left(E\left[X_{t+1}\left|\mathcal{F}_t\right.\right]\right)\right), \\ \widehat{P}_{t+1|t+1} &= \left(I - K_{t+1}H_{t+1}\right)\widehat{P}_{t+1|t},\end{aligned}$$

where

$$\begin{aligned}K_{t+1} &= \widehat{P}_{t+1|t} H_{t+1}'\left(H_{t+1}\widehat{P}_{t+1|t}H_{t+1}' + \omega I\right)^{-1}, \\ H_{t+1} &= \left(\left.\frac{\partial G\left(X_{t+1}\right)}{\partial X_{t+1}}\right|_{\widehat{X}_{t+1|t}}\right)' = \left[G'\left(E\left[X_{t+1}\left|\mathcal{F}_t\right.\right]\right) \quad 0_{J\times n_X}\right].\end{aligned}$$

In deriving the expression for H_{t+1} we have used the result $G'(z) = \Phi(z)$.

Maximum Likelihood Estimation. The parameters are estimated by maximizing the log likelihood, which is given by

$$
\begin{aligned}
\mathcal{L} &= -\frac{(J+n_M)T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log\left|\widehat{V}_{t|t-1}\right| \\
&\quad -\frac{1}{2}\sum_{t=1}^{T}\left(Y_t-\widehat{Y}_{t|t-1}\right)'\left(\widehat{V}_{t|t-1}\right)^{-1}\left(Y_t-\widehat{Y}_{t|t-1}\right).
\end{aligned}
$$

D.2. State Space Representation under the MFSRTSM

The state-space representation of the MFSRTSM is almost identical as that of the MFSRTSM0, described in the previous subsection, except that forward-rate approximation depends on both X_{t+1} and M_{t+1}:

$$F^o_{t+1} = G\left(X_{t+1}, M_{t+1}\right) + \eta_{t+1},$$

where

$$
G\left(X_{t+1}, M_{t+1}\right) = \begin{bmatrix} \underline{r} + \sigma^{\mathbb{Q}}_{n_1} g\left(\frac{a_{n_1}+b'_{n_1}X_t-\underline{r}}{\sigma^{\mathbb{Q}}_{n_1}}, \frac{c_{n_1}+d'_{n_1}M_t+e'_{n_1}X_t-\underline{m}}{\tau^{\mathbb{Q}}_{n_1}}; \varrho_n\right) \\ \vdots \\ \underline{r} + \sigma^{\mathbb{Q}}_{n_J} g\left(\frac{a_{n_J}+b'_{n_J}X_t-\underline{r}}{\sigma^{\mathbb{Q}}_{n_J}}, \frac{c_{n_J}+d'_{n_J}M_t+e'_{n_J}X_t-\underline{m}}{\tau^{\mathbb{Q}}_{n_J}}; \varrho_n\right) \end{bmatrix}.
$$

The extended Kalman filter is implemented similarly as the MFSRTSM0, except that the Kalman gain is now given by

$$
\begin{aligned}
K_{t+1} &= \widehat{P}_{t+1|t}H'_{t+1}\left(H_{t+1}\widehat{P}_{t+1|t}H'_{t+1}+\omega I\right)^{-1}, \\
H_{t+1} &= \left(\left.\frac{\partial G\left(X_{t+1}, M_{t+1}\right)}{\partial X_{t+1}}\right|_{\widehat{X}_{t+1|t}}\right)'.
\end{aligned}
$$

From Lemma 9 below, we can show that H_{t+1} is a $J \times n_X$ matrix and its i_{th} row is given by $\left[F\left(x^i_1, x^i_2; \varrho\right)b'_{n_i} + h\left(-x^i_2, -x^i_1; \varrho\right)\left[x^i_1 - \varrho x^i_2 + \sqrt{1-\varrho^2}\right]e'_{n_i} \quad 0_{1\times n_X}\right]$, where $x^i_1 = \frac{a_{n_i}+b'_{n_i}E[X_{t+1}|\mathcal{F}_t]-\underline{r}}{\sigma^{\mathbb{Q}}_{n_i}}$, and $x^i_2 = \frac{c_{n_i}+d'_{n_i}M_{t+1}+e'_{n_i}X_t-\underline{m}}{\tau^{\mathbb{Q}}_{n_i}}$.

Based on the extended Kalam filter, the log likelihood can be calculated similarly as the MFSRTSM0.

Lemma 9. *Suppose $g\left(x_1, x_2; \varrho\right)$ is defined by equation (C-3), then the following results hold:*

$$
\begin{aligned}
\frac{\partial g\left(x_1, x_2; \varrho\right)}{\partial x_1} &= F\left(x_1, x_2; \varrho\right), \\
\frac{\partial g\left(x_1, x_2; \varrho\right)}{\partial x_2} &= h\left(-x_2, -x_1; \varrho\right)\left(x_1 - \varrho x_2 + \sqrt{1-\varrho^2}\right),
\end{aligned}
$$

where functions $F\left(\cdot, \cdot; \varrho\right)$ and $h\left(\cdot, \cdot; \varrho\right)$ are defined in Lemma 5.

Proof. By definition, we have

$$\begin{aligned}
\frac{\partial g\left(x_1,x_2;\varrho\right)}{\partial x_1} &= -\left[x_1 h\left(-x_1,-x_2;\varrho\right)+\frac{\varrho}{\sqrt{1-\varrho^2}}\phi\left(-x_1\right)\phi\left(\frac{-\varrho x_1+x_2}{\sqrt{1-\varrho^2}}\right)\right], \\
&\quad -\varrho\left[-\frac{1}{\sqrt{1-\varrho^2}}\phi\left(-x_2\right)\phi\left(\frac{-\varrho x_2+x_1}{\sqrt{1-\varrho^2}}\right)\right]+h\left(-x_1,-x_2;\varrho\right)x_1+F\left(x_1,x_2;\varrho\right), \\
&= \frac{\varrho}{\sqrt{1-\varrho^2}}\left[\phi\left(x_2\right)\phi\left(\frac{x_1-\varrho x_2}{\sqrt{1-\varrho^2}}\right)-\phi\left(x_1\right)\phi\left(\frac{\varrho x_1-x_2}{\sqrt{1-\varrho^2}}\right)\right]+F\left(x_1,x_2;\varrho\right), \\
&= F\left(x_1,x_2;\varrho\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial G\left(x_1,x_2\right)}{\partial x_2} \\
&= \frac{1}{\sqrt{1-\varrho^2}}\phi\left(-x_1\right)\phi\left(\frac{-\varrho x_1+x_2}{\sqrt{1-\varrho^2}}\right)+h\left(-x_2,-x_1;\varrho\right)x_1 \\
&\quad -\varrho\left[x_2 h\left(-x_2,-x_1;\varrho\right)+\frac{\varrho}{\sqrt{1-\varrho^2}}\phi\left(-x_2\right)\phi\left(\frac{-\varrho x_2+x_1}{\sqrt{1-\varrho^2}}\right)\right] \\
&= h\left(-x_2,-x_1;\varrho\right)\left[x_1-\varrho x_2\right] \\
&\quad +\frac{1}{\sqrt{1-\varrho^2}}\left[\phi\left(x_1\right)\phi\left(\frac{-\varrho x_1+x_2}{\sqrt{1-\varrho^2}}\right)-\varrho^2\phi\left(x_2\right)\phi\left(\frac{-\varrho x_2+x_1}{\sqrt{1-\varrho^2}}\right)\right] \\
&= h\left(-x_2,-x_1;\varrho\right)\left(x_1-\varrho x_2\right)+\frac{1-\varrho^2}{\sqrt{1-\varrho^2}}h\left(-x_2,-x_1;\varrho\right), \\
&= h\left(-x_2,-x_1;\varrho\right)\left(x_1-\varrho x_2+\sqrt{1-\varrho^2}\right).
\end{aligned}$$

□

E Macroeconomic Data

ID	MNEMONIC	DESCRIPTION	Transformation
		Group 1: NIPA	
1	GDPC1*	Real Gross Domestic Product, 3 Decimal (BIL. CHAIN 2012 $)	Δln
2	PCECC96*	Real Personal Consumption Expenditures (BIL. CHAIN 2012 $)	Δln
3	PCDGx*	Real personal consumption expenditures: Durable goods (BIL. CHAIN 2012 $)	Δln
4	PCESVx*	Real Personal Consumption Expenditures: Services (BIL. 2012 $)	Δln
5	PCNDx*	Real Personal Consumption Expenditures: Nondurable Goods (BIL. 2012 $)	Δln
6	GPDIC1*	Real Gross Private Domestic Investment, 3 decimal (BIL. CHAIN 2012 $)	Δln
7	FPIx*	Real private fixed investment (BIL. CHAIN 2012 $)	Δln
8	Y033RC1Q027SBEAx*	Real Gross Private Domestic Investment: Nonresidential (BIL. CHAIN 2012 $)	Δln
9	PNFIx*	Real private fixed investment: Nonresidential (BIL. CHAIN 2012 $)	Δln
10	PRFIx*	Real private fixed investment: Residential (BIL. CHAIN 2012 $)	Δln
11	A014RE1Q156NBEA*	Shares of gross domestic product: Change in private inventories (Percent)	
12	GCEC1*	Real Government Expenditures and Investment (BIL. CHAIN 2012 $)	Δln
13	A823RL1Q225SBEA*	Real Government Expenditures and Investment: Federal (Percent Change)	
14	FGRECPTx*	Real Federal Government Current Receipts (BIL. CHAIN 2012 $)	Δln
15	SLCEx*	Real government state and local consumption expenditures (BIL. CHAIN 2012 $)	Δln
16	EXPGSC1*	Real Exports of Goods and Services, 3 Decimal (BIL. CHAIN 2012 $)	Δln
17	IMPGSC1*	Real Imports of Goods and Services, 3 Decimal (BIL. CHAIN 2012 $)	Δln
18	DPIC96*	Real Disposable Personal Income (BIL. CHAIN 2012 $)	Δln
19	OUTNFB*	Nonfarm Business Sector: Real Output (2009=100)	Δln
20	OUTBS*	Business Sector: Real Output (2009=100)	Δln
21	OUTMS*	Manufacturing Sector: Real Output (2009=100)	Δln
188	B020RE1Q156NBEA*	Shares of gross domestic product: Exports of goods and services (Percent)	
189	B021RE1Q156NBEA*	Shares of gross domestic product: Imports of goods and services (Percent)	
		Group 2: Industrial Production	
22	INDPRO*	Industrial Production Index (2012=100)	Δln
23	IPFINAL*	Industrial Production: Final Products (Market Group) (2012=100)	Δln
24	IPCONGD*	Industrial Production: Consumer Goods (2012=100)	Δln
25	IPMAT*	Industrial Production: Materials (2012=100)	Δln
26	IPDMAT*	Industrial Production: Durable Materials (2012=100)	Δln
27	IPNMAT*	Industrial Production: Nondurable Materials (2012=100)	Δln
28	IPDCONGD*	Industrial Production: Durable Consumer Goods (2012=100)	Δln
29	IPB51110SQ*	Industrial Production: Durable Goods: Automotive products (2012=100)	Δln
30	IPNCONGD*	Industrial Production: Nondurable Consumer Goods (2012=100)	Δln
31	IPBUSEQ*	Industrial Production: Business Equipment (2012=100)	Δln
32	IPB51220SQ*	Industrial Production: Consumer energy products (2012=100)	Δln
33	TCU*	Capacity Utilization: Total Industry (Percent of Capacity)	
34	CUMFNS*	Capacity Utilization: Manufacturing (SIC) (Percent of Capacity)	
192	IPMANSICS*	Industrial Production: Manufacturing (SIC) (2012=100)	Δln
193	IPB51222S*	Industrial Production: Residential Utilities (2012=100)	Δln
194	IPFUELS*	Industrial Production: Fuels (2012=100)	Δln
		Group 3: Employment and Unemployment	
35	PAYEMS*	All Employees: Total nonfarm (Thousands of Persons)	Δln
36	USPRIV*	All Employees: Total Private Industries (Thousands of Persons)	Δln
37	MANEMP*	All Employees: Manufacturing (Thousands of Persons)	Δln
38	SRVPRD*	All Employees: Service-Providing Industries (Thousands of Persons)	Δln
39	USGOOD*	All Employees: Goods-Producing Industries (Thousands of Persons)	Δln
40	DMANEMP*	All Employees: Durable goods (Thousands of Persons)	Δln
41	NDMANEMP*	All Employees: Nondurable goods (Thousands of Persons)	Δln
42	USCONS*	All Employees: Construction (Thousands of Persons)	Δln
43	USEHS*	All Employees: Education and Health Services (Thousands of Persons)	Δln
44	USFIRE*	All Employees: Financial Activities (Thousands of Persons)	Δln
45	USINFO*	All Employees: Information Services (Thousands of Persons)	Δln
46	USPBS*	All Employees: Professional and Business Services (Thousands of Persons)	Δln
47	USLAH*	All Employees: Leisure and Hospitality (Thousands of Persons)	Δln
48	USSERV*	All Employees: Other Services (Thousands of Persons)	Δln
49	USMINE*	All Employees: Mining and logging (Thousands of Persons)	Δln
50	USTPU*	All Employees: Trade, Transportation and Utilities (Thousands of Persons)	Δln
51	USGOVT*	All Employees: Government (Thousands of Persons)	Δln
52	USTRADE*	All Employees: Retail Trade (Thousands of Persons)	Δln
53	USWTRADE*	All Employees: Wholesale Trade (Thousands of Persons)	Δln
54	CES9091000001*	All Employees: Government: Federal (Thousands of Persons)	Δln
55	CES9092000001*	All Employees: Government: State Government (Thousands of Persons)	Δln
56	CES9093000001*	All Employees: Government: Local Government (Thousands of Persons)	Δln
57	CE16OV*	Civilian Employment (Thousands of Persons)	Δln
58	CIVPART*	Civilian Labor Force Participation Rate (Percent)	
59	UNRATE*	Civilian Unemployment Rate (Percent)	
60	UNRATESTx*	Unemployment Rate less than 27 weeks (Percent)	
61	UNRATELTx*	Unemployment Rate for more than 27 weeks (Percent)	
62	LNS14000012*	Unemployment Rate - 16 to 19 years (Percent)	
63	LNS14000025*	Unemployment Rate - 20 years and over, Men (Percent)	
64	LNS14000026*	Unemployment Rate - 20 years and over, Women (Percent)	
65	UEMPLT5*	Number of Civilians Unemployed - Less Than 5 Weeks (Thousands of Persons)	Δln
66	UEMP5TO14*	Number of Civilians Unemployed for 5 to 14 Weeks (Thousands of Persons)	Δln
67	UEMP15T26*	Number of Civilians Unemployed for 15 to 26 Weeks (Thousands of Persons)	Δln
68	UEMP27OV*	Number of Civilians Unemployed for 27 Weeks and Over (Thousands of Persons)	Δln
69	LNS13023621*	Unemployment Level - Job Losers (Thousands of Persons)	Δln
70	LNS13023557*	Unemployment Level - Reentrants to Labor Force (Thousands of Persons)	Δln
71	LNS13023705*	Unemployment Level - Job Leavers (Thousands of Persons)	Δln
72	LNS13023569*	Unemployment Level - New Entrants (Thousands of Persons)	Δln
73	LNS12032194*	Employment Level - Part-Time for Economic Reasons, All Industries (Thousands of Persons)	Δln
74	HOABS*	Business Sector: Hours of All Persons (2009=100)	Δln
75	HOAMS*	Manufacturing Sector: Hours of All Persons (2009=100)	Δln
76	HOANBS*	Nonfarm Business Sector: Hours of All Persons (2009=100)	Δln
77	AWHMAN*	AVG WKLY Hours of Production Employees: Manufacturing (Hours)	
78	AWHNONAG*	AVG WKLY Hours Of Production Employees: Total private (Hours)	
79	AWOTMAN*	AVG WKLY Overtime Hours of Production Employees: Manufacturing (Hours)	
80	HWIx*	Help-Wanted Index	Δln
195	UEMPMEAN*	Average (Mean) Duration of Unemployment (Weeks)	
196	CES0600000007*	AVG WKLY Hours of Production Employees: Goods-Producing	
218	HWIURATIOx*	Ratio of Help Wanted/No. Unemployed	
219	CLAIMSx*	Initial Claims	Δln

ID	MNEMONIC	DESCRIPTION	Transformation
		Group 4: Housing	
81	HOUST	Housing Starts: Total: New Privately Owned Housing Units Started (Thousands of Units)	ln
82	HOUST5F	Privately Owned Housing Starts: 5-Unit Structures or More (Thousands of Units)	ln
83	PERMIT	New Private Housing Units Authorized by Building Permits (Thousands of Units)	ln
84	HOUSTMW	Housing Starts in Midwest Census Region (Thousands of Units)	ln
85	HOUSTNE	Housing Starts in Northeast Census Region (Thousands of Units)	ln
86	HOUSTS	Housing Starts in South Census Region (Thousands of Units)	ln
87	HOUSTW	Housing Starts in West Census Region (Thousands of Units)	ln
177	USSTHPI	All-Transactions House Price Index for the United States (1980 Q1=100)	Δln
178	SPCS10RSA	S&P/Case-Shiller 10-City Composite Home Price Index (January 2000 = 100)	Δln
179	SPCS20RSA	S&P/Case-Shiller 20-City Composite Home Price Index (January 2000 = 100)	Δln
225	PERMITNE	New Private Housing Units Authorized by Building Permits: Northeast (THOUS.,SA)	ln
226	PERMITMW	New Private Housing Units Authorized by Building Permits: Midwest (THOUS.,SA)	ln
227	PERMITS	New Private Housing Units Authorized by Building Permits: South (THOUS.,SA)	ln
228	PERMITW	New Private Housing Units Authorized by Building Permits: West(THOUS.,SA)	ln
		Group 5: Inventories, Orders, and Sales	
88	CMRMTSPLx	Real Manufacturing and Trade Industries Sales (MIL. CHAIN 2012 $)	Δln
89	RSAFSx	Real Retail and Food Services Sales (MIL. CHAIN 2012 $)	Δln
90	AMDMNOx	Real Manufacturers' New Orders: Durable Goods (Millions of 2012 Dollars)	Δln
91	ACOGNOx	Real Value of Manufacturers' New Orders for Consumer Goods Industries (MIL. 2012 $)	Δln
92	AMDMUOx	Real Value of Manufacturers' Unfilled Orders for Durable Goods Industries (MIL. 2012 $)	Δln
93	ANDENOx	Real Value of Manufacturers' New Orders for Capital Goods (MIL. 2012 $)	Δln
94	INVCQRMTSPL	Real Manufacturing and Trade Inventories (Millions of 2012 Dollars)	Δln
220	BUSINVx*	Total Business Inventories (MIL. $)	Δln
221	ISRATIOx*	Total Business: Inventories to Sales Ratio	
		Group 6: Prices	
95	PCECTPI*	Personal Consumption Expenditures: Chain-type Price Index (2009=100)	Δln
96	PCEPILFE*	Personal Consumption Expenditures Excluding Food and Energy (2009=100)	Δln
97	GDPCTPI*	Gross Domestic Product: Chain-type Price Index (2009=100)	Δln
98	GPDICTPI*	Gross Private Domestic Investment: Chain-type Price Index (2009=100)	Δln
99	IPDBS*	Business Sector: Implicit Price Deflator (2009=100)	Δln
100	DGDSRG3Q086SBEA*	Personal consumption expenditures: Goods	Δln
101	DDURRG3Q086SBEA*	Personal consumption expenditures: Durable goods	Δln
102	DSERRG3Q086SBEA*	Personal consumption expenditures: Services	Δln
103	DNDGRG3Q086SBEA*	Personal consumption expenditures: Nondurable goods	Δln
104	DHCERG3Q086SBEA*	Personal consumption expenditures: Services: Household consumption expenditures	Δln
105	DMOTRG3Q086SBEA*	Personal consumption expenditures: Durable goods: Motor vehicles and parts	Δln
106	DFDHRG3Q086SBEA*	Personal consumption expenditures: Durable goods: Furnishings	Δln
107	DREQRG3Q086SBEA*	Personal consumption expenditures: Durable goods: Recreational	Δln
108	DODGRG3Q086SBEA*	Personal consumption expenditures: Durable goods: Other durable goods	Δln
109	DFXARG3Q086SBEA*	Personal consumption expenditures: Nondurable goods: Food and beverages	Δln
110	DCLORG3Q086SBEA*	Personal consumption expenditures: Nondurable goods: Clothing and footwear	Δln
111	DGOERG3Q086SBEA*	Personal consumption expenditures: Nondurable goods: Gasoline and other energy goods	Δln
112	DONGRG3Q086SBEA*	Personal consumption expenditures: Nondurable goods: Other nondurable goods	Δln
113	DHUTRG3Q086SBEA*	Personal consumption expenditures: Services: Housing and utilities	Δln
114	DHLCRG3Q086SBEA*	Personal consumption expenditures: Services: Health care	Δln
115	DTRSRG3Q086SBEA*	Personal consumption expenditures: Transportation services	Δln
116	DRCARG3Q086SBEA*	Personal consumption expenditures: Recreation services	Δln
117	DFSARG3Q086SBEA*	Personal consumption expenditures: Services: Food services and accommodations	Δln
118	DIFSRG3Q086SBEA*	Personal consumption expenditures: Financial services and insurance	Δln
119	DOTSRG3Q086SBEA*	Personal consumption expenditures: Other services	Δln
120	CPIAUCSL*	CPI-U: All Items (1982-84=100)	Δln
121	CPILFESL*	CPI-U: All Items Less Food and Energy (1982-84=100)	Δln
122	WPSFD49207*	Producer Price Index: Finished Goods (1982=100)	Δln
123	PPIACO*	Producer Price Index for All Commodities (1982=100)	Δln
124	WPSFD49502*	Producer Price Index: Finished Consumer Goods (1982=100)	Δln
125	WPSFD4111*	Producer Price Index: Finished Consumer Foods (1982=100)	Δln
126	PPIIDC*	Producer Price Index: Industrial Commodities (1982=100)	Δln
127	WPSID61*	Producer Price Index: Intermediate Materials: Supplies and Components (1982=100)	Δln
128	WPU0531*	Producer Price Index: Fuels and Related Products and Power: Natural Gas (1982=100)	Δln
129	WPU0561*	Producer Price Index: Fuels and Related Products and Power: Crude Petroleum (1982=100)	Δln
130	OILPRICEx	Real Crude Oil Prices: West Texas Intermediate (WTI) - Cushing, Oklahoma (2012 $/Barrel)	Δln
203	WPSID62*	Producer Price Index: Crude Materials for Further Processing (1982=100)	Δln
204	PPICMM*	Producer Price Index: Commodities: Metals and metal products (1982=100)	Δln
205	CPIAPPSL*	CPI-U: Apparel (1982-84=100)	Δln
206	CPITRNSL*	CPI-U: Transportation (1982-84=100)	Δln
207	CPIMEDSL*	CPI-U: Medical Care (1982-84=100)	Δln
208	CUSR0000SAC*	CPI-U: Commodities (1982-84=100)	Δln
209	CUSR0000SAD*	CPI-U: Durables (1982-84=100)	Δln
210	*	CPI-U: Services (1982-84=100)	Δln
211	CPIULFSL*	CPI-U: All Items Less Food (1982-84=100)	Δln
212	CUSR0000SA0L2*	CPI-U: All items less shelter (1982-84=100)	Δln
213	CUSR0000SA0L5*	CPI-U: All items less medical care (1982-84=100)	Δln
231	CUSR0000SEHC*	CPI for All Urban Consumers: Owners' equivalent rent of residences (Dec 1982=100)	Δln
		Group 7: Earnings and Productivity	
131	AHETPIx*	Real AVG HRLY Earnings of Production Employees: Total Private (2012 $/Hour)	Δln
132	CES2000000008x*	Real AVG HRLY Earnings of Production Employees: Construction (2012 $/Hour)	Δln
133	CES3000000008x*	Real AVG HRLY Earnings of Production Employees: Manufacturing (2012 $/Hour)	Δln
134	COMPRMS*	Manufacturing Sector: Real Compensation Per Hour (2009=100)	Δln
135	COMPRNFB*	Nonfarm Business Sector: Real Compensation Per Hour (2009=100)	Δln
136	RCPHBS*	Business Sector: Real Compensation Per Hour (2009=100)	Δln
137	OPHMFG*	Manufacturing Sector: Real Output Per Hour of All Persons (2009=100)	Δln
138	OPHNFB*	Nonfarm Business Sector: Real Output Per Hour of All Persons (2009=100)	Δln
139	OPHPBS*	Business Sector: Real Output Per Hour of All Persons (2009=100)	Δln
140	ULCBS*	Business Sector: Unit Labor Cost (2009=100)	Δln
141	ULCMFG*	Manufacturing Sector: Unit Labor Cost (2009=100)	Δln
142	ULCNFB*	Nonfarm Business Sector: Unit Labor Cost (2009=100)	Δln
143	UNLPNBS*	Nonfarm Business Sector: Unit Nonlabor Payments (2009=100)	Δln
214	CES0600000008*	AVG HRLY Earnings of Production Employees: Goods-Producing ($/Hour)	Δln

ID	MNEMONIC	DESCRIPTION	Transformation
		Group 8: Interest Rates	
144	FEDFUNDS	Effective Federal Funds Rate (Percent)	
145	TB3MS	3-Month Treasury Bill: Secondary Market Rate (Percent)	
146	TB6MS	6-Month Treasury Bill: Secondary Market Rate (Percent)	
147	GS1	1-Year Treasury Constant Maturity Rate (Percent)	
148	GS10	10-Year Treasury Constant Maturity Rate (Percent)	
149	MORTGAGE30US	30-Year Conventional Mortgage Rate (Percent)	
150	AAA	Moody's Seasoned Aaa Corporate Bond Yield (Percent)	
151	BAA	Moody's Seasoned Baa Corporate Bond Yield (Percent)	
152	BAA10YM	Moody's Seasoned Baa Corporate Bond Yield Relative to 10-Year Treasury Yield (Percent)	
153	MORTG10YRx	30-Year Conventional Mortgage Rate Relative to 10-Year Treasury Constant Maturity (Percent)	
154	TB6M3Mx	6-Month Treasury Bill Minus 3-Month Treasury Bill, secondary market (Percent)	
155	GS1TB3Mx	1-Year Treasury Constant Maturity Minus 3-Month Treasury Bill, secondary market (Percent)	
156	GS10TB3Mx	10-Year Treasury Constant Maturity Minus 3-Month Treasury Bill, secondary market (Percent)	
157	CPF3MTB3Mx	3-Month Commercial Paper Minus 3-Month Treasury Bill, secondary market (Percent)	
199	GS5	5-Year Treasury Constant Maturity Rate	
200	TB3SMFFM	3-Month Treasury Constant Maturity Minus Federal Funds Rate	
201	T5YFFM	5-Year Treasury Constant Maturity Minus Federal Funds Rate	
202	AAAFFM	Moody's Seasoned Aaa Corporate Bond Minus Federal Funds Rate	
223	CP3M	3-Month AA Financial Commercial Paper Rate	
224	COMPAPFF	3-Month Commercial Paper Minus Federal Funds Rate	
		Group 9: Money and Credit	
158	BOGMBASEREALx	St. Louis Adjusted Monetary Base (Billions of 1982-84 Dollars)	Δln
159	M1REAL	Real M1 Money Stock (Billions of 1982-84 Dollars)	Δln
160	M2REAL	Real M2 Money Stock (Billions of 1982-84 Dollars)	Δln
161	BUSLOANSx	Real Commercial and Industrial Loans, All Commercial Banks (Billions of 2009 U.S. Dollars)	Δln
162	CONSUMERx	Real Consumer Loans at All Commercial Banks (Billions of 2009 U.S. Dollars)	Δln
163	NONREVSLx	Total Real Nonrevolving Credit Owned and Securitized, Outstanding (Billions of Dollars)	Δln
164	REALLNx	Real Real Estate Loans, All Commercial Banks (Billions of 2009 U.S. Dollars)	Δln
165	REVOLSLx	Total Real Revolving Credit Owned and Securitized, Outstanding (BIL. 2012 $)	Δln
166	TOTALSLx	Total Consumer Credit Outstanding	Δln
167	DRIWCIL	FRBSLOO: Net Percentage of Respondents More Willing to Make Consumer Loans	
197	TOTRESNS	Total Reserves of Depository Institutions (Billions of Dollars)	
198	NONBORRES	Reserves Of Depository Institutions, Nonborrowed (MIL. $)	
215	DTCOLNVHFNM	Consumer Motor Vehicle Loans Outstanding Owned by Finance Companies (MIL. $)	Δln
216	DTCTHFNM	Total Consumer Loans Owned and Securitized by Finance Companies (MIL. $)	Δln
217	INVEST	Securities in Bank Credit at All Commercial Banks (Billions of Dollars)	Δln
		Group 10: Household Balance Sheets	
168	TABSHNOx*	Real Total Assets of Households(BIL. 2012 $)	Δln
169	TLBSHNOx*	Real Total Liabilities of Households(BIL. 2012 $)	Δln
170	LIABPIx*	Liabilities of HouseholdsRelative to Personal Disposable Income (Percent)	
171	TNWBSHNOx*	Real Net Worth of Households(BIL. 2012 $)	Δln
172	NWPIx*	Net Worth of HouseholdsRelative to Disposable Personal Income (Percent)	
173	TARESAx*	Real Assets of Householdsexcluding Real Estate Assets (BIL. 2012 $)	Δln
174	HNOREMQ027S*x	Real Real Estate Assets of Households(BIL. 2012 $)	Δln
175	TFAABSHNOx	Real Total Financial Assets of Households(BIL. 2012 $)	Δln
222	CONSPIx	Nonrevolving consumer credit to Personal Income	
		Group 11: Exchange Rates	
180	TWEXAFEGSMTHx	Trade Weighted U.S. Dollar Index: Major Currencies (March 1973=100)	Δln
181	EXUSEU	U.S. / Euro Foreign Exchange Rate (U.S. Dollars to One Euro)	Δln
182	EXSZUSx	Switzerland / U.S. Foreign Exchange Rate	Δln
183	EXJPUSx	Japan / U.S. Foreign Exchange Rate	Δln
184	EXUSUKx	U.S. / U.K. Foreign Exchange Rate	Δln
185	EXCAUSx	Canada / U.S. Foreign Exchange Rate	Δln
		Group 12: Other	
186	UMCSENTx	University of Michigan: Consumer Sentiment (1st Quarter 1966=100)	Δln
187	USEPUINDXM	Economic Policy Uncertainty Index for United States	Δln
		Group 13: Stock Markets	
176	VIXCLSx	CBOE Volatility Index: VIX	
229	NIKKEI225	Nikkei Stock Average	Δln
230	NASDAQCOM	NASDAQ Composite (Feb 5, 1971=100)	Δln
243	S&P 500	S&P's Common Stock Price Index: Composite	Δln
244	S&P: indust	S&P's Common Stock Price Index: Industrials	Δln
245	S&P div yield	S&P's Composite Common Stock: Dividend Yield	
246	S&P PE ratio	S&P's Composite Common Stock: Price-Earnings Ratio	
		Group 14: Non-Household Balance Sheets	
190	GFDEGDQ188S*	Federal Debt: Total Public Debt as Percent of GDP (Percent)	
191	GFDEBTNx*	Real Federal Debt: Total Public Debt (Millions of 2012 Dollars)	Δln
232	TLBSNNCBx*	Real Nonfinancial Corporate Business Sector Liabilities (BIL. 2012 $)	Δln
233	TLBSNNCBBDIx*	Nonfinancial Corporate Business Sector Liabilities to Disposable Business Income (Percent)	
234	TTAABSNNCBx*	Real Nonfinancial Corporate Business Sector Assets (BIL. 2012 $)	Δln
235	TNWMVBSNNCBx*	Real Nonfinancial Corporate Business Sector Net Worth (BIL. 2012 $)	Δln
236	TNWMVBSNNCBBDIx*	Nonfinancial Corporate Business Sector Net Worth to Disposable Business Income (Percent)	Δln
237	TLBSNNBx*	Real Nonfinancial Noncorporate Business Sector Liabilities (BIL. 2012 $)	Δln
238	TLBSNNBBDIx*	Nonfinancial Noncorporate Business Sector Liabilities to Disposable Business Income (Percent)	Δln
239	TABSNNBx*	Real Nonfinancial Noncorporate Business Sector Assets (BIL. 2012 $)	Δln
240	TNWBSNNBx*	Real Nonfinancial Noncorporate Business Sector Net Worth (BIL. 2012 $)	Δln
241	TNWBSNNBBDIx*	Nonfinancial Noncorporate Business Sector Net Worth to Disposable Business Income (Percent)	
242	CNCFx	Real Disposable Business Income, BIL. 2012 $	Δln